Chapter 5

Preface to Curvature

5.8 Exercises

1

(a) Repeat the argument leading to Equation 5.1, but this time assume that only a fraction $\epsilon < 1$ of the mass's kinetic energy is converted into a photon.

If only a fraction ϵ of the energy is converted into a photon, then it will start with an energy of $\epsilon(m + mgh + \mathcal{O}(\boldsymbol{v}^4)$, but once it reaches the top it should have an energy of ϵm, as it loses the component due to gravitational potential energy. Thus

$$\frac{E'}{E} = \frac{\epsilon m}{\epsilon(m + mgh + \mathcal{O}(\boldsymbol{v}^4))} = \frac{m}{m + mgh + \mathcal{O}(\boldsymbol{v}^4)} = 1 - gh + \mathcal{O}\Big(\boldsymbol{v}^4\Big)$$

(b) Assume Equation 5.1 does not hold. Devise a perpetual motion device.

If we assume that the photon does not return to an energy m once it reaches the top, but instead has an energy $m' > m$, then we could create the perpetual motion device shown in Figure 5.1. A black box consumes the photon with energy m', and splits it into a new object of mass m, and a photon of energy $m' - m$. The object repeats the action of the original falling mass, creating an infinite loop.

2 Explain why a uniform gravitational field would not be able to create tides on Earth.

Tides depend on there being a gravitational field gradient. If the curvature closer to the source of the field (e.g. the Moon) is greater than it is further away, then the closer side will move towards the source more than the further side, thus creating tides. In the absense of such a gradient, there would be no difference in curvature between the two sides, and thus they would not stretch relative to each other.

7 Calculate the components of $\Lambda^{\alpha'}{}_{\beta}$ and $\Lambda^{\mu}{}_{\nu'}$ for transformations $(x, y) \leftrightarrow (r, \theta)$.

$$\begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial \theta/\partial x & \partial \theta/\partial y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta \theta \end{pmatrix}$$

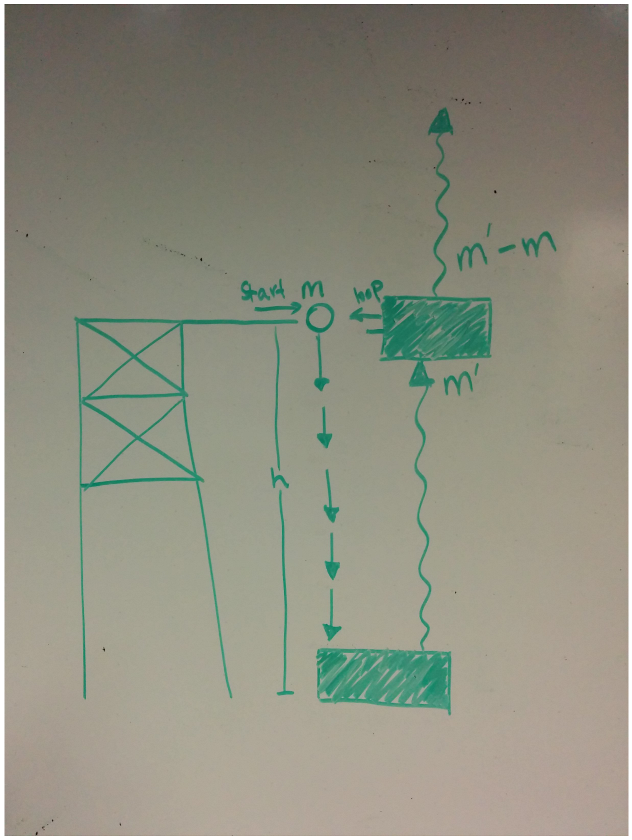

Figure 5.1: Problem 1: Perpetual motion device.

$$= \begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad = \begin{pmatrix} x/\sqrt{x^2+y^2} & -y \\ y/\sqrt{x^2+y^2} & x \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix}$$

$$\Lambda^r{}_x = x/\sqrt{x^2+y^2} = \cos\theta \qquad \Lambda^x{}_r = \cos\theta = x/\sqrt{x^2+y^2}$$

$$\Lambda^r{}_y = y/\sqrt{x^2+y^2} = \sin\theta \qquad \Lambda^y{}_r = \sin\theta = y/\sqrt{x^2+y^2}$$

$$\Lambda^\theta{}_x = -y/(x^2+y^2) = -(1/r)\sin\theta \qquad \Lambda^x{}_\theta = -r\sin\theta = -y$$

$$\Lambda^\theta{}_y = x/(x^2+y^2) = (1/r)\cos\theta \qquad \Lambda^y{}_\theta = r\cos\theta = x$$

8

(a) $f \equiv x^2+y^2+2xy$, $\vec{V} \underset{(x,y)}{\rightarrow} (x^2+3y, y^2+3x)$, $\vec{W} \underset{(r,\theta)}{\rightarrow} (1,1)$. Express $f = f(r,\theta)$, and find the components of $\vec{V}$ and $\vec{W}$ in a polar basis, as functions of r and θ.

$$f = x^2+y^2+2xy = (x+y)^2$$

$$= (r\cos\theta + r\sin\theta)^2 = r^2\sin^2\theta + r^2\cos^2\theta + 2r^2\sin\theta\cos\theta$$
$$= r^2(1+\sin(2\theta))$$

$$\vec{V} \underset{(x,y)}{\rightarrow} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix}$$

$$\vec{V} \underset{(r,\theta)}{\rightarrow} \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix}$$

$$\underset{(r,\theta)}{\rightarrow} \begin{pmatrix} r^2\cos^2\theta + 6r\sin\theta\cos\theta + r^2\sin^3\theta \\ -r\cos^2\theta\sin\theta - 3\sin^2\theta + r\sin^2\theta\cos\theta + 3\cos^2\theta \end{pmatrix}$$

$$\underset{(r,\theta)}{\rightarrow} \begin{pmatrix} r^2(\sin^3\theta + \cos^3\theta) + 6r\sin\theta\cos\theta \\ r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix}$$

$$\underset{(r,\theta)}{\rightarrow} \begin{pmatrix} r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta) \\ (r/2)\sin(2\theta)(\sin\theta - \cos\theta) + 3\cos(2\theta) \end{pmatrix}$$

$$\vec{W} \underset{(r,\theta)}{\rightarrow} \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underset{(r,\theta)}{\rightarrow} \begin{pmatrix} \cos\theta + \sin\theta \\ (1/r)(\cos\theta - \sin\theta) \end{pmatrix}$$

(b) Express the components of $\tilde{\mathrm{d}}f$ in (x,y) and obtain them in (r,θ) by:

(i) using direct calculation in (r,θ):

$$\tilde{\mathrm{d}}f \underset{(r,\theta)}{\rightarrow} \big(\partial f/\partial r\ ,\ \partial f/\partial\theta\big) = \Big(2r(1+\sin(2\theta)), 2r^2\cos(2\theta)\Big)$$

(ii) transforming the components in (x,y):

$$\tilde{\mathrm{d}}f \underset{(x,y)}{\rightarrow} \big(\partial f/\partial x\ ,\ \partial f/\partial y\big) = \big(2(x+y), 2(x+y)\big) = \big(2r(\cos\theta+\sin\theta), 2r(\cos\theta+\sin\theta)\big)$$

$$\Big((\tilde{\mathrm{d}}f)_r \quad (\tilde{\mathrm{d}}f)_\theta\Big) = \Big(1 \quad 1\Big) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \big[2r(\cos\theta + \sin\theta)\big]$$
$$= \Big(2r(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta) \quad 2r^2(\cos^2\theta - \sin^2\theta)\Big)$$
$$= \Big(2r(1+\sin(2\theta)) \quad 2r^2\cos(2\theta)\Big)$$

(c) Now find the (r,θ) components of the one-forms $\tilde{V}$ and $\tilde{W}$ associated with the vectors $\vec{V}$ and $\vec{W}$ by

(i) using the metric tensor in (r,θ):

$$
\begin{aligned}
V_r &= g_{r\alpha}V^\alpha = g_{rr}V^r + g_{r\theta}V^\theta \\
&= r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta) \\
V_\theta &= g_{\theta r}V^r + g_{\theta\theta}V^\theta = (1/2)r^3\sin(2\theta)(\sin\theta - \cos\theta) + 3r^2\cos(2\theta) \\
W_r &= g_{r\alpha}W^\alpha = g_{rx}W^x + g_{ry}W^y \\
&= 1(\cos\theta + \sin\theta) + 0\big[(1/r)(\cos\theta - \sin\theta)\big] \\
&= \cos\theta + \sin\theta \\
W_\theta &= g_{\theta x}W^x + g_{\theta y}W^y = \\
&= 0(\cos\theta + \sin\theta) + r^2\big[r(\cos\theta - \sin\theta)\big] \\
&= r(\cos\theta - \sin\theta)
\end{aligned}
$$

(ii) using the metric tensor in (x, y) and then doing a coordinate transformation:

$$
\begin{aligned}
V_x &= V^x; \quad V_y = V^y \\
V_r &= \Lambda^\alpha{}_r V_\alpha = \Lambda^x{}_r V_x + \Lambda^y{}_r V_y \\
&= \cos\theta V_x + \sin\theta V_y \\
&= r^2\cos^3\theta + (3/2)r\sin(2\theta) + r^2\sin^3\theta + (3/2)r\sin(2\theta) \\
&= r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta) \\
V_\theta &= \Lambda^\alpha{}_\theta V_\alpha = \Lambda^x{}_\theta V_x + \Lambda^y{}_\theta V_y \\
&= (-r\sin\theta)V_x + (r\cos\theta)V_y \\
&= -r^3\cos^2\theta\sin\theta - 3r^2\sin^2\theta + r^3\sin^2\theta\cos\theta + 3r^2\cos^2\theta \\
&= r^3\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r^2(\cos^2\theta - \sin^2\theta) \\
&= (1/2)r^3\sin(2\theta)(\sin\theta - \cos\theta) + 3r^2\cos(2\theta) \\
W_x &= W^x = W_y = W^y = 1 \\
W_r &= \Lambda^\alpha{}_r W_\alpha = \Lambda^x{}_r W_x + \Lambda^y{}_r W_y \\
&= \cos\theta + \sin\theta \\
W_\theta &= \Lambda^\alpha{}_\theta W_\alpha = \Lambda^x{}_\theta W_x + \Lambda^y{}_\theta W_y \\
&= -r\sin\theta + r\cos\theta \\
&= r(\cos\theta - \sin\theta)
\end{aligned}
$$

11 Consider $V \underset{(x,y)}{\to} (x^2 + 3y, y^2 + 3x)$.

(a) Find $V^{\alpha}{}_{,\beta}$ in Cartesian coordinates.

$$V^x{}_{,x} = 2x; \quad V^y{}_{,y} = 2y; \quad V^x{}_{,y} = V^y{}_{,x} = 3.$$

(b)

$$\begin{aligned}
V^{\mu'}{}_{;\nu'} &= \Lambda^{\mu'}{}_{\alpha}\Lambda^{\beta}{}_{\nu'}V^{\alpha}{}_{,\beta} \\
V^r{}_{;r} &= \Lambda^r{}_x\Lambda^x{}_rV^x{}_{,x} + \Lambda^r{}_y\Lambda^y{}_rV^y{}_{,y} + \Lambda^r{}_x\Lambda^y{}_rV^x{}_{,y} + \Lambda^r{}_y\Lambda^x{}_rV^y{}_{,x} \\
&= (\cos^2\theta)(2r\cos\theta) + (\sin^2\theta)(2r\sin\theta) + (\sin\theta\cos\theta)(3) + (\sin\theta\cos\theta)(3) \\
&= 2r(\cos^3\theta + \sin^3\theta) + 3\sin(2\theta) \\
V^{\theta}{}_{;\theta} &= \Lambda^{\theta}{}_x\Lambda^{\theta}{}_rV^x{}_{,x} + \Lambda^{\theta}{}_y\Lambda^{\theta}{}_rV^y{}_{,y} + \Lambda^{\theta}{}_x\Lambda^y{}_{\theta}V^x{}_{,y} + \Lambda^{\theta}{}_y\Lambda^{\theta}{}_rV^y{}_{,x} \\
&= (\sin^2\theta)(2r\cos\theta) + (\cos^2\theta)(2r\sin\theta) + (-\sin\theta\cos\theta)(3) + (-\sin\theta\cos\theta)(3) \\
&= \sin(2\theta)[r(\sin\theta + \cos\theta) - 3] \\
V^r{}_{;\theta} &= \Lambda^r{}_x\Lambda^x{}_{\theta}V^x{}_{,x} + \Lambda^r{}_y\Lambda^y{}_{\theta}V^y{}_{,y} + \Lambda^r{}_x\Lambda^y{}_{\theta}V^x{}_{,y} + \Lambda^r{}_y\Lambda^x{}_{\theta}V^y{}_{,x} \\
&= (-r\sin\theta\cos\theta)(2r\cos\theta) + (r\sin\theta\cos\theta)(2r\sin\theta) + (r\cos^2\theta)(3) + (-r\sin^2\theta) \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta) \\
V^{\theta}{}_{;r} &= \Lambda^{\theta}{}_x\Lambda^x{}_rV^x{}_{,x} + \Lambda^{\theta}{}_y\Lambda^y{}_rV^y{}_{,y} + \Lambda^{\theta}{}_x\Lambda^y{}_rV^x{}_{,y} + \Lambda^{\theta}{}_y\Lambda^x{}_rV^y{}_{,x} \\
&= (-(1/r)\sin\theta\cos\theta)(2r\cos\theta) + ((1/r)\sin\theta\cos\theta)(2r\sin\theta) + (-(1/r)\sin^2\theta)(3) + ((1/r)\cos^2\theta)(3) \\
&= \sin(2\theta)(\sin\theta - \cos\theta) + \frac{3}{r}\cos(2\theta)
\end{aligned}$$

(c) compute $V^{\mu'}{}_{;\nu'}$ directly in polars using the Christoffel symbols.
Recall that we have $\Gamma^{\mu}{}_{rr} = \Gamma^r{}_{r\theta} = \Gamma^{\theta}{}_{\theta\theta} = 0$, $\Gamma^{\theta}{}_{r\theta} = 1/r$, and $\Gamma^r{}_{\theta\theta} = -r$.

$$\begin{aligned}
V^{\mu'}{}_{;\nu'} &= V^{\mu'}{}_{,\nu'} + V^{\alpha'}\Gamma^{\mu'}{}_{\alpha'\nu'} \\
V^r{}_{;r} &= V^r{}_{,r} + V^{\alpha'}\Gamma^r{}_{\alpha'r} \\
V^r{}_{,r} &= \partial V^r/\partial r = 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
V^{\alpha}\Gamma^r{}_{\alpha r} &= V^r\Gamma^r{}_{rr} + V^{\theta}\Gamma^r{}_{\theta r} = 0 \\
V^r{}_{;r} &= V^r{}_{,r} = 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
V^{\theta}{}_{;\theta} &= V^{\theta}{}_{,\theta} + V^{\alpha'}\Gamma^{\theta}{}_{\alpha'\theta} \\
V^{\theta}{}_{,\theta} &= \partial V^{\theta}\big/\partial\theta = (r/2)\sin(2\theta)(\sin\theta + \cos\theta) + r\cos(2\theta)(\sin\theta - \cos\theta) - 6\sin(2\theta) \\
V^{\alpha'}\Gamma^{\theta}{}_{\alpha'\theta} &= V^r\Gamma^{\theta}{}_{r\theta} + V^{\theta}\Gamma^{\theta}{}_{\theta\theta} \\
&= [r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta)](1/r) \\
&= r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta)
\end{aligned}$$

$$
\begin{aligned}
V^{\theta}{}_{;\theta} &= \sin(2\theta)[r(\sin\theta + \cos\theta) - 3] \\
V^{r}{}_{;\theta} &= V^{r}{}_{,\theta} + V^{r}\Gamma^{r}{}_{r\theta} + V^{\theta}\Gamma^{r}{}_{\theta\theta} \\
&= V^{r}{}_{,\theta} + V^{\theta}\Gamma^{r}{}_{\theta\theta} = \partial V^{r}/\partial\theta - rV^{\theta} \\
&= 6r\cos(2\theta) + (3/2)r^2\sin(2\theta)(\sin\theta - \cos\theta) - ((1/2)r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta)) \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta) \\
V^{\theta}{}_{;r} &= V^{\theta}{}_{,r} + V^{r}\Gamma^{\theta}{}_{rr} + V^{\theta}\Gamma^{\theta}{}_{\theta r} = V^{\theta}{}_{,r} + \frac{1}{r}V^{\theta} \\
&= (1/2)\sin(2\theta)(\sin\theta - \cos\theta) + (1/2)\sin(2\theta)(\sin\theta - \cos\theta) + (3/r)\cos(2\theta) \\
&= \sin(2\theta)(\sin\theta - \cos\theta) + (3/r)\cos(2\theta)
\end{aligned}
$$

(d) Calculate the divergence using the results from part (a)

$$V^{\alpha}{}_{,\alpha} = V^{x}{}_{,x} + V^{y}{}_{,y} = 2(x + y) = 2r(\sin\theta + \cos\theta)$$

(e) Calculate the divergence using the results from either part (b) or (c).

$$
\begin{aligned}
V^{\mu'}{}_{;\mu'} &= V^{r}{}_{;r} + V^{\theta}{}_{;\theta} \\
&= 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) + \sin(2\theta)[r(\sin\theta + \cos\theta) - 3] \\
&= 2r(\sin\theta + \cos\theta)
\end{aligned}
$$

(f) Compute $V^{\mu'}{}_{;\mu'}$ using Equation 5.56.

$$V^{\mu'}{}_{;\mu'} = \frac{1}{r}\frac{\partial}{\partial r}(rV^{r}) + \frac{\partial}{\partial\theta}(V^{\theta}) = 2r(\sin\theta + \cos\theta)$$

12

$$\tilde{p} \underset{(x,y)}{\rightarrow} (x^2 + 3y, y^2 + 3x).$$

(a) Find the components $p_{\alpha,\beta}$ in Cartesian coordinates.

Since $p_{\alpha,\beta} = \partial p_{\alpha}/\partial x^{\beta}$, it's simply $p_{x,x} = 2x$, $p_{y,y} = 2y$, and $p_{x,y} = p_{y,x} = 3$.

(b) Find the components $p_{\mu';\nu'}$ in polar coordinates by using the transformation $\Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}p_{\alpha,\beta}$.

$$
\begin{aligned}
p_{r;r} &= (\Lambda^{x}{}_{r})^2 p_{x,x} + (\Lambda^{y}{}_{r})^2 p_{y,y} + 2\Lambda^{x}{}_{r}\Lambda^{y}{}_{r}p_{x,y} \\
&= (\cos^2\theta)(2r\cos\theta) + (\sin^2\theta)(2r\sin\theta) + 2(\sin\theta\cos\theta)(3) \\
&= 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
p_{\theta;\theta} &= (\Lambda^{x}{}_{\theta})^2 p_{x,x} + \left(\Lambda^{y}{}_{\theta}\right)^2 p_{y,y} + 2\Lambda^{x}{}_{\theta}\Lambda^{y}{}_{\theta}p_{x,y} \\
&= (-r\sin\theta)^2(2r\cos\theta) + (r\cos\theta)^2(2r\sin\theta) + 2(3(-r\sin\theta)(r\cos\theta)) \\
&= r^2\sin(2\theta)(r(\sin\theta + \cos\theta) - 3)
\end{aligned}
$$

$$\begin{aligned}
p_{r;\theta} &= \Lambda^x{}_r\Lambda^x{}_\theta p_{x,x} + \Lambda^y{}_r\Lambda^y{}_\theta p_{y,y} + \Lambda^x{}_r\Lambda^y{}_\theta p_{x,y} + \Lambda^y{}_r\Lambda^x{}_\theta p_{y,x} \\
&= (-r\sin\theta\cos\theta)(2r\cos\theta) + (r\sin\theta\cos\theta)(2r\sin\theta) + 3(r\cos^2\theta - r\sin^2\theta) \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta),
\end{aligned}$$

and by the symmetry of $p_{\alpha,\beta}$ in Cartesian coordinates, $p_{\theta;r} = p_{r;\theta}$.

(c) Now find $p_{\mu';\nu'}$ using the Christoffel symbols.

$$\begin{aligned}
p_{r;r} &= p_{r,r} - p_r\Gamma^r{}_{rr} - p_\theta\Gamma^\theta{}_{rr} = p_{r,r} = \partial p_r/\partial r \\
&= \partial/\partial r\left[r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta)\right] = 2r(sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
p_{\theta;\theta} &= p_{\theta,\theta} - p_r\Gamma^r{}_{\theta\theta} - p_\theta\Gamma^\theta{}_{\theta\theta} = p_{\theta,\theta} + rp_r = \partial p_\theta/\partial\theta \\
&= \partial/\partial\theta\left[(1/2)r^3\sin(2\theta)(\sin\theta - \cos\theta) + 3r^2\cos(2\theta)\right] + r\left[r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta)\right] \\
&= r^2\sin(2\theta)\left[r(\sin\theta + \cos\theta) - 3\right] \\
p_{r;\theta} &= p_{r,\theta} - p_r\Gamma^r{}_{r\theta} - p_\theta\Gamma^\theta{}_{r\theta} = \partial p_r/\partial\theta - (1/r)p_\theta \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta) \\
p_{\theta;r} &= p_{\theta,r} - p_r\Gamma^r{}_{\theta r} - p_\theta\Gamma^\theta{}_{\theta r} = \partial p_\theta/\partial r - (1/r)p_\theta \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta)
\end{aligned}$$

13 Show in polars that $g_{\mu'\alpha'}V^{\alpha'}{}_{;\nu'} = p_{\mu';\nu'}$.

$$\begin{aligned}
g_{r\alpha'}V^{\alpha'}{}_{;r} &= g_{rr}V^r{}_{;r} + g_{r\theta}V^\theta{}_{;r} \\
&= 1V^r{}_{;r} = p_{r;r} \\
g_{\theta\alpha'}V^{\alpha'}{}_{;\theta} &= g_{\theta r}V^r{}_{;\theta} + g_{\theta\theta}V^\theta{}_{;\theta} \\
&= r^2V^\theta{}_{;\theta} = p_{\theta;\theta} \\
g_{r\alpha'}V^{\alpha'}{}_{;\theta} &= g_{rr}V^r{}_{;\theta} + g_{r\theta}V^\theta{}_{;\theta} \\
&= 1V^r{}_{;\theta} = p_{\theta;r} \\
g_{\theta\alpha'}V^{\alpha'}{}_{;r} &= g_{\theta r}V^r{}_{;r} + g_{\theta\theta}V^\theta{}_{;r} \\
&= r^2V^\theta{}_{;r} = p_{\theta;r}
\end{aligned}$$

14 Compute $\boldsymbol{\nabla}_\beta A^{\mu\nu}$ for the tensor $\boldsymbol{A}$ with components:

$$A^{rr} = r^2, \qquad A^{r\theta} = r\sin\theta,$$

$$A^{\theta\theta} = \tan\theta, \qquad A^{\theta r} = r\cos\theta$$

$$
\begin{aligned}
A^{rr}{}_{,r} &= 2r & A^{rr}{}_{,\theta} &= 0 \\
A^{\theta\theta}{}_{,r} &= 0 & A^{\theta\theta}{}_{,\theta} &= \sec^2\theta \\
A^{r\theta}{}_{,r} &= \sin\theta & A^{r\theta}{}_{,\theta} &= r\cos\theta \\
A^{\theta r}{}_{,r} &= \cos\theta & A^{\theta r}{}_{,\theta} &= -r\sin\theta
\end{aligned}
$$

$$\boldsymbol{\nabla}_\beta A^{\mu\nu} = A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu}\Gamma^\mu{}_{\alpha\beta} + A^{\mu\alpha}\Gamma^\nu{}_{\alpha\beta}$$

$$
\begin{aligned}
\boldsymbol{\nabla}_r A^{rr} &= A^{rr}{}_{,r} + A^{\alpha r}\Gamma^r{}_{\alpha r} + A^{r\alpha}\Gamma^r{}_{\alpha r} \\
&= A^{rr}{}_{,r} + A^{rr}\Gamma^r{}_{rr} + A^{\theta r}\Gamma^r{}_{\theta r} + A^{rr}\Gamma^r{}_{rr} + A^{r\theta}\Gamma^r{}_{\theta r} \\
&= A^{rr}{}_{,r} = 2r \\
\boldsymbol{\nabla}_\theta A^{rr} &= A^{rr}{}_{,\theta} + A^{\alpha r}\Gamma^r{}_{\alpha\theta} + A^{r\alpha}\Gamma^r{}_{\alpha\theta} \\
&= A^{rr}{}_{,\theta} + A^{rr}\Gamma^r{}_{r\theta} + A^{\theta r}\Gamma^r{}_{\theta\theta} + A^{rr}\Gamma^r{}_{r\theta} + A^{r\theta}\Gamma^r{}_{\theta\theta} \\
&= (A^{\theta r} + A^{r\theta})\Gamma^r{}_{\theta\theta} = -r^2(\sin\theta + \cos\theta) \\
\boldsymbol{\nabla}_r A^{\theta\theta} &= A^{\theta\theta}{}_{,r} + A^{\alpha\theta}\Gamma^\theta{}_{\alpha r} + A^{\theta\alpha}\Gamma^\theta{}_{\alpha r} \\
&= A^{\theta\theta}{}_{,r} + A^{r\theta}\Gamma^\theta{}_{rr} + A^{\theta\theta}\Gamma^\theta{}_{\theta r} + A^{\theta r}\Gamma^\theta{}_{rr} + A^{\theta\theta}\Gamma^\theta{}_{\theta r} \\
&= 2A^{\theta\theta}\Gamma^\theta{}_{\theta r} = (2/r)\tan\theta \\
\boldsymbol{\nabla}_\theta A^{\theta\theta} &= A^{\theta\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^\theta{}_{\alpha\theta} + A^{\theta\alpha}\Gamma^\theta{}_{\alpha\theta} \\
&= A^{\theta\theta}{}_{,\theta} + A^{r\theta}\Gamma^\theta{}_{r\theta} + A^{\theta\theta}\Gamma^\theta{}_{\theta\theta} + A^{\theta r}\Gamma^\theta{}_{r\theta} + A^{\theta\theta}\Gamma^\theta{}_{\theta\theta} \\
&= A^{\theta\theta}{}_{,\theta} + (A^{r\theta} + A^{\theta r})\Gamma^\theta{}_{r\theta} = \sin\theta + \cos\theta + \sec^2\theta \\
\boldsymbol{\nabla}_r A^{r\theta} &= A^{r\theta}{}_{,r} + A^{\alpha\theta}\Gamma^r{}_{\alpha r} + A^{r\alpha}\Gamma^\theta{}_{\alpha r} \\
&= A^{r\theta}{}_{,r} + A^{r\theta}\Gamma^r{}_{rr} + A^{\theta\theta}\Gamma^r{}_{\theta r} + A^{rr}\Gamma^\theta{}_{rr} + A^{r\theta}\Gamma^\theta{}_{\theta r} \\
&= A^{r\theta}{}_{,r} + A^{r\theta}\Gamma^\theta{}_{\theta r} = 2\sin\theta \\
\boldsymbol{\nabla}_\theta A^{r\theta} &= A^{r\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^r{}_{\alpha\theta} + A^{r\alpha}\Gamma^\theta{}_{\alpha\theta} \\
&= A^{r\theta}{}_{,\theta} + A^{r\theta}\Gamma^r{}_{r\theta} + A^{\theta\theta}\Gamma^r{}_{\theta\theta} + A^{rr}\Gamma^\theta{}_{r\theta} + A^{r\theta}\Gamma^\theta{}_{\theta\theta} \\
&= A^{r\theta}{}_{,\theta} + A^{\theta\theta}\Gamma^r{}_{\theta\theta} + A^{rr}\Gamma^\theta{}_{r\theta} = r(1 + \cos\theta - \tan\theta) \\
\boldsymbol{\nabla}_r A^{\theta r} &= A^{\theta r}{}_{,r} + A^{\alpha r}\Gamma^\theta{}_{\alpha r} + A^{\theta\alpha}\Gamma^r{}_{\alpha r} \\
&= A^{\theta r}{}_{,r} + A^{rr}\Gamma^\theta{}_{rr} + A^{\theta r}\Gamma^\theta{}_{\theta r} + A^{\theta r}\Gamma^r{}_{rr} + A^{\theta\theta}\Gamma^r{}_{\theta r} \\
&= A^{\theta r}{}_{,r} + A^{\theta r}\Gamma^\theta{}_{\theta r} = 2\cos\theta \\
\boldsymbol{\nabla}_\theta A^{\theta r} &= A^{\theta r}{}_{,\theta} + A^{\alpha r}\Gamma^\theta{}_{\alpha\theta} + A^{\theta\alpha}\Gamma^r{}_{\alpha\theta} \\
&= A^{\theta r}{}_{,\theta} + A^{rr}\Gamma^\theta{}_{r\theta} + A^{\theta r}\Gamma^\theta{}_{\theta\theta} + A^{\theta r}\Gamma^r{}_{r\theta} + A^{\theta\theta}\Gamma^r{}_{\theta\theta} \\
&= A^{\theta r}{}_{,\theta} + A^{rr}\Gamma^\theta{}_{r\theta} + A^{\theta\theta}\Gamma^r{}_{\theta\theta} = -r\sin\theta
\end{aligned}
$$

15 Find the components of $V^\alpha{}_{;\beta;\gamma}$ for the vector $V^r = 1$, $V^\theta = 0$.

We start by finding the components of $V^\alpha{}_{;\beta}$.

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}.$$

By noting that $V^\alpha{}_{,\beta} = V^\theta = \Gamma^r{}_{rr} = \Gamma^r{}_{r\theta} = 0$, we can simplify this to

$$V^\alpha{}_{;\beta} = V^r \Gamma^\alpha{}_{r\beta},$$

which means

$$V^r{}_{;r} = V^r{}_{;\theta} = V^\theta{}_{;r} = 0; \quad V^\theta{}_{;\theta} = \frac{1}{r}.$$

Now we can say

$$V^\alpha{}_{;\beta;\mu} = \boldsymbol{\nabla}_\mu V^\alpha{}_{;\beta} = V^\alpha{}_{;\beta,\mu} + V^\gamma{}_{;\beta} \Gamma^\alpha{}_{\gamma\mu} - V^\alpha{}_{;\gamma} \Gamma^\gamma{}_{\beta\mu}.$$

Note that $V^\theta{}_{;\theta}$ is a function only of r, and so $V^\theta{}_{;\theta,r} = -1/r^2$, and all other partial derivatives are zero. We can also see by inspecting the components, that $V^r{}_{;\mu;\nu} = V^\theta{}_{;\mu} \Gamma^r{}_{\theta\mu}$, as all other components go to zero. Likewise, we can see that $V^\theta{}_{;r;\mu} = -V^\theta{}_{;\theta} \Gamma^\theta{}_{r\mu}$. It then becomes easy to find all the individual components. I summarize their values in Table 5.1.

16 Repeat the steps leading from Equation 5.74 to 5.75.

Recalling that $g_{\alpha\mu;\beta} = 0$, we can rewrite Equation 5.72 as

$$g_{\alpha\beta,\mu} = \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu}.$$

Now if we switch the β and μ indices, and then switch the α and β indices, we get two more equations,

$$g_{\alpha\mu,\beta} = \Gamma^\nu{}_{\alpha\beta} g_{\nu\mu} + \Gamma^\nu{}_{\mu\beta} g_{\alpha\nu},$$
$$g_{\beta\mu,\alpha} = \Gamma^\nu{}_{\beta\alpha} g_{\nu\mu} + \Gamma^\nu{}_{\mu\alpha} g_{\beta\nu}.$$

Now we add the first two equations and subtract the third, getting

$$\begin{aligned} g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} + \Gamma^\nu{}_{\alpha\beta} g_{\nu\mu} + \Gamma^\nu{}_{\mu\beta} g_{\alpha\nu} - \Gamma^\nu{}_{\beta\alpha} g_{\nu\mu} - \Gamma^\nu{}_{\mu\alpha} g_{\beta\nu} \\ &= \Gamma^\nu{}_{\alpha\mu} g_{\beta\nu} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} + \Gamma^\nu{}_{\alpha\beta} g_{\nu\mu} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} - \Gamma^\nu{}_{\alpha\beta} g_{\nu\mu} - \Gamma^\nu{}_{\alpha\mu} g_{\beta\nu} \end{aligned}$$

α	β	μ	$V^\alpha{}_{;\beta;\mu}$
θ	θ	θ	0
θ	θ	r	$-1/r^2$
θ	r	θ	$-1/r^2$
θ	r	r	0
r	θ	θ	-1
r	θ	r	0
r	r	θ	0
r	r	r	0

Table 5.1: Components of the tensor in Exercise 15.

$$= 2\Gamma^{\nu}{}_{\beta\mu} g_{\alpha\nu}.$$

Recalling that $g^{\alpha\gamma} g_{\alpha\nu} = g^{\gamma}{}_{\nu} = \delta^{\gamma}{}_{\nu}$, we divide both sides by 2 and multiply by $g^{\alpha\gamma}$, arriving at Equation 5.75:

$$\begin{aligned}\frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \frac{2}{2} g^{\alpha\gamma} g_{\alpha\nu} \Gamma^{\nu}{}_{\beta\mu} \\ &= \Gamma^{\nu}{}_{\beta\mu}\end{aligned}$$

17 Show how $V^{\beta}{}_{,\alpha}$ and $V^{\mu}\Gamma^{\beta}{}_{\upsilon\alpha}$ transform under change of coordinates. Neither follows a tensor transformation law, but their *sum* does.

$$\begin{aligned}
V^{\alpha'}{}_{,\beta'} &= \frac{\partial V^{\alpha'}}{\partial x^{\beta'}} = \Lambda^{\beta}{}_{\beta'} \frac{\partial}{\partial x^{\beta}} \left[\Lambda^{\alpha'}{}_{\alpha} V^{\alpha} \right] \\
&= \Gamma^{\beta}{}_{\beta'} \left[V^{\alpha} \frac{\partial}{\partial x^{\beta}} \Lambda^{\alpha'}{}_{\alpha} + \Lambda^{\alpha'}{}_{\alpha} \frac{\partial}{\partial x^{\beta}} V^{\alpha} \right] \\
&= \Lambda^{\beta}{}_{\beta'} V^{\alpha} \Lambda^{\alpha'}{}_{\alpha,\beta} + \Lambda^{\beta}{}_{\beta'} \Lambda^{\alpha'}{}_{\alpha} V^{\alpha}{}_{,\beta} \\
&\neq \Lambda^{\beta}{}_{\beta'} \Lambda^{\alpha'}{}_{\alpha} V^{\alpha}{}_{,\beta} \\
\frac{\partial \vec{e}_{\alpha'}}{\partial x^{\beta'}} &= \Lambda^{\beta}{}_{\beta'} \frac{\partial}{\partial x^{\beta}} [\Lambda^{\alpha}{}_{\alpha'} \vec{e}_{\alpha}] \\
&= \Lambda^{\beta}{}_{\beta'} \left[\Lambda^{\alpha}{}_{\alpha'} \frac{\partial}{\partial x^{\beta}} \vec{e}_{\alpha} + \vec{e}_{\alpha} \frac{\partial}{\partial x^{\beta}} \Lambda^{\alpha}{}_{\alpha'} \right] \\
&= \Lambda^{\beta}{}_{\beta'} \Lambda^{\alpha}{}_{\alpha'} \Gamma^{\mu}{}_{\alpha\beta} \vec{e}_{\mu} + \Lambda^{\beta}{}_{\beta'} \Lambda^{\alpha}{}_{\alpha',\beta} \vec{e_{\alpha}} \\
&\neq \Lambda^{\beta}{}_{\beta'} \Lambda^{\alpha}{}_{\alpha'} \Gamma^{\mu}{}_{\alpha\beta} \vec{e}_{\mu},
\end{aligned}$$

so we have shown that $\partial \vec{e}_{\alpha'} \Big/ \partial x^{\beta'}$ is not a tensor, and since V^{μ} is a tensor, and the product of a tensor and a non-tensor is also not a tensor, then $V^{\mu}\Gamma^{\beta}{}_{\nu\alpha}$ is not a tensor.
According to Carroll, the precise transformation is

$$\Gamma^{\nu'}{}_{\mu'\lambda'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\lambda}{}_{\lambda'} \Lambda^{\nu'}{}_{\nu} \Gamma^{\nu}{}_{\mu\lambda} + \Lambda^{\mu}{}_{\mu'} \Lambda^{\lambda}{}_{\lambda'} \Lambda^{\nu'}{}_{\mu\lambda}.$$

Now we add the two expressions, in order to show that it is a tensor equation

$$\begin{aligned}
V^{\nu'}{}_{,\lambda'} + V^{\mu'} \Gamma^{\nu'}{}_{\mu'\lambda'} &= \Lambda^{\lambda}{}_{\lambda'} V^{\nu} \Lambda^{\nu'}{}_{\nu,\lambda} + \Lambda^{\lambda}{}_{\lambda'} \Lambda^{\nu'}{}_{\nu} V^{\nu}{}_{,\lambda} + \Lambda^{\lambda}{}_{\lambda'} \Lambda^{\nu'}{}_{\nu} V^{\mu} \Gamma^{\nu}{}_{\mu\lambda} + \Lambda^{\lambda}{}_{\lambda'} V^{\nu} \Lambda^{\nu'}{}_{\lambda,\mu} \\
&= \Lambda^{\lambda}{}_{\lambda'} \Lambda^{\nu'}{}_{\nu} \left(V^{\nu}{}_{,\lambda} + V^{\mu} \Gamma^{\nu}{}_{\mu\lambda} \right)
\end{aligned}$$

So it does in fact transform like a tensor equation, meaning $V^{\nu}{}_{;\lambda}$ is a tensor!
18

Verify Equation 5.78:

$$\left.\begin{aligned} \vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} \equiv g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}} \\ \tilde{\omega}^{\hat{\alpha}} \cdot \tilde{\omega}^{\hat{\beta}} \equiv g^{\hat{\alpha}\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}} \end{aligned}\right\}$$

For the basis *vectors*, we have

$$
\begin{aligned}
g_{\hat{r}\hat{r}} &= \vec{e}_{\hat{r}} \cdot \vec{e}_{\hat{r}} = \vec{e}_r \cdot \vec{e}_r = g_{rr} = 1 \\
g_{\hat{\theta}\hat{\theta}} &= \vec{e}_{\hat{\theta}} \cdot \vec{e}_{\hat{\theta}} = \left(\frac{1}{r}\vec{e}_\theta\right) \cdot \left(\frac{1}{r}\vec{e}_\theta\right) = \frac{1}{r^2}(\vec{e}_\theta \cdot \vec{e}_\theta) = \frac{1}{r} g_{r\theta} = 1 \\
g_{\hat{r}\hat{\theta}} &= \vec{e}_{\hat{r}} \cdot \vec{e}_{\hat{\theta}} = \vec{e}_r \cdot \left(\frac{1}{r}\vec{e}_\theta\right) = \frac{1}{r}(\vec{e}_r \cdot \vec{e}_\theta) = \frac{1}{r} g_{r\theta} = 0 \\
g_{\hat{\theta}\hat{r}} &= g_{\hat{r}\hat{\theta}} = 0
\end{aligned}
$$

So it is indeed true that $g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}}$.

Now for the basis *one-forms*, we have

$$
\begin{aligned}
g^{\hat{r}\hat{r}} &= \tilde{\omega}^{\hat{r}} \cdot \tilde{\omega}^{\hat{r}} = \tilde{\mathrm{d}}r \cdot \tilde{\mathrm{d}}r = g^{rr} = 1 \\
g^{\hat{\theta}\hat{\theta}} &= \tilde{\omega}^{\hat{\theta}} \cdot \tilde{\omega}^{\hat{\theta}} = (r\tilde{\mathrm{d}}\theta) \cdot (r\tilde{\mathrm{d}}\theta) = r^2(\tilde{\mathrm{d}}\theta \cdot \tilde{\mathrm{d}}\theta) = r^2 g^{\theta\theta} = r^2(1/r^2) = 1 \\
g^{\hat{r}\hat{\theta}} &= \tilde{\omega}^{\hat{r}} \cdot \tilde{\omega}^{\hat{\theta}} = \tilde{\mathrm{d}}r \cdot (r\tilde{\mathrm{d}}\theta) = r(\tilde{\mathrm{d}}r \cdot \tilde{\mathrm{d}}\theta) = r g^{r\theta} = 0 \\
g^{\hat{\theta}\hat{r}} &= g^{\hat{r}\hat{\theta}} = 0
\end{aligned}
$$

So it is indeed true that $g^{\hat{\alpha}\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}}$.

19 Repeat the calculations going from Equations 5.81 to 5.84, with $\tilde{\mathrm{d}}r$ and $\tilde{\mathrm{d}}\theta$ as your bases. Show that they form a coordinate basis.

$$
\begin{aligned}
\tilde{\mathrm{d}}r &= \cos\theta \,\mathrm{d}x + \sin\theta \,\mathrm{d}y = \frac{\partial \xi}{\partial x}\tilde{\mathrm{d}}x + \frac{\partial \xi}{\partial y}\tilde{\mathrm{d}}y \\
\frac{\partial \xi}{\partial x} &= \cos\theta; \quad \frac{\partial \xi}{\partial y} = \sin\theta \\
\frac{\partial}{\partial y}\frac{\partial \xi}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \xi}{\partial y} \implies \frac{\partial}{\partial y}(x/r) = \frac{\partial}{\partial x}(y/r),
\end{aligned}
$$

which is true, so we have shown that at least $\tilde{\mathrm{d}}r$ may be part of a coordinate basis.

$$
\begin{aligned}
\tilde{\mathrm{d}}\theta &= -\frac{1}{r}\sin\theta\tilde{\mathrm{d}}x + \frac{1}{r}\cos\theta\tilde{\mathrm{d}}y = \frac{\partial \eta}{\partial x}\tilde{\mathrm{d}}x + \frac{\partial \eta}{\partial y}\tilde{\mathrm{d}}y \\
\frac{\partial \eta}{\partial x} &= -\frac{1}{r}\sin\theta; \quad \frac{\partial \eta}{\partial y} = \frac{1}{r}\cos\theta \\
\frac{\partial}{\partial y}\frac{\partial \eta}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \eta}{\partial y} \implies \frac{\partial}{\partial y}\left[-\frac{1}{r}\sin\theta\right] = \frac{\partial}{\partial x}\left[\frac{1}{r}\cos\theta\right],
\end{aligned}
$$

which is also true, and thus we have shown that $\tilde{\mathrm{d}}r$ and $\tilde{\mathrm{d}}\theta$ form a coordinate basis.

20 For a non-coordinate basis $\{\vec{e}_\mu\}$, let $c^\alpha{}_{\mu\nu} = \boldsymbol{\nabla}_{\vec{e}_\mu}\vec{e}_\nu - \boldsymbol{\nabla}_{\vec{e}_\nu}\vec{e}_\mu$. Use this in place of Equation 5.74 to derive a more general expression for Equation 5.75.

$\boldsymbol{c}$ is antisymmetric w.r.t. its bottom indices.

$$
c^\alpha{}_{\mu\nu}\vec{e}_\alpha + c^\alpha{}_{\nu\mu}\vec{e}_\alpha = (\boldsymbol{\nabla}_{\vec{e}_\mu}\vec{e}_\nu - \boldsymbol{\nabla}_{\vec{e}_\nu}\vec{e}_\mu) + (\boldsymbol{\nabla}_{\vec{e}_\nu}\vec{e}_\mu - \boldsymbol{\nabla}_{\vec{e}_\mu}\vec{e}_\nu) = 0
$$

$$\implies c^{\alpha}{}_{\mu\nu}\vec{e}_{\alpha} = -c^{\alpha}{}_{\nu\mu}\vec{e}_{\alpha}$$
$$\implies c^{\alpha}{}_{\mu\nu} = -c^{\alpha}{}_{\nu\mu}$$

Expanding the covariant derivatives in the original expression, we get

$$\begin{aligned}
c^{\alpha}{}_{\mu\nu}\vec{e}_{\alpha} &= \vec{e}_{\nu;\mu} - \vec{e}_{\mu;\nu} \\
&= (\vec{e}_{\nu,\mu} - \vec{e}_{\alpha}\Gamma^{\alpha}{}_{\nu\mu}) - (\vec{e}_{\mu,\nu} - \vec{e}_{\alpha}\Gamma^{\alpha}{}_{\mu\nu}) \\
&= \vec{e}_{\alpha}(\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}) \\
c^{\alpha}{}_{\mu\nu} &= \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}
\end{aligned}$$

Now we recall the result from Exercise 16, but without assuming symmetry of the Christoffel symbols

$$\begin{aligned}
g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \Gamma^{\nu}{}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta}g_{\alpha\nu} - \Gamma^{\nu}{}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}{}_{\mu\alpha}g_{\beta\nu} \\
&= \Gamma^{\nu}{}_{\alpha\mu}g_{\beta\nu} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta}g_{\alpha\nu} - \Gamma^{\nu}{}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}{}_{\mu\alpha}g_{\beta\nu} \\
&= g_{\beta\nu}(\Gamma^{\nu}{}_{\alpha\mu} - \Gamma^{\nu}{}_{\mu\alpha}) + g_{\alpha\nu}(\Gamma^{\nu}{}_{\beta\mu} + \Gamma^{\nu}{}_{\mu\beta}) + g_{\nu\mu}(\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\beta\alpha}) \\
&= g_{\beta\nu}c^{\nu}{}_{\alpha\mu} + g_{\alpha\nu}(\Gamma^{\nu}{}_{\beta\mu} + \Gamma^{\nu}{}_{\mu\beta} + \Gamma^{\nu}{}_{\beta\mu} - \Gamma^{\nu}{}_{\beta\mu}) + g_{\nu\mu}c^{\nu}{}_{\alpha\beta} \\
&= g_{\beta\nu}c^{\nu}{}_{\alpha\mu} + g_{\nu\mu}c^{\nu}{}_{\alpha\beta} + g_{\alpha\nu}(2\Gamma^{\nu}{}_{\beta\mu} + c^{\nu}{}_{\mu\beta}) \\
g^{\nu\mu}2g_{\alpha\nu}\Gamma^{\nu}{}_{\beta\mu} &= g^{\nu\mu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c_{\beta\alpha\mu} - c_{\mu\alpha\beta} - c_{\alpha\mu\beta}) \\
\Gamma^{\nu}{}_{\beta\alpha} &= \frac{1}{2}g^{\nu\mu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c_{\beta\alpha\mu} - c_{\mu\alpha\beta} - c_{\alpha\mu\beta})
\end{aligned}$$

21 A uniformly accelerated observer has world line

$$t(\lambda) = a\sinh\lambda, \quad x(\lambda) = a\cosh\lambda$$

(a) Show that the spacelike line tangent to his world line (which is parameterized by λ) is orthogonal to the line parameterized by a.

The line tangent to his world line is

$$\vec{V} \to \frac{\mathrm{d}}{\mathrm{d}\lambda}(t,x) = (a\cosh\lambda, a\sinh\lambda).$$

The line parameterized by a is

$$\vec{W} \to \frac{\mathrm{d}}{\mathrm{d}a}(t,x) = (\sinh\lambda, \cosh\lambda)$$

If they are orthogonal, then their dot product must be zero

$$\vec{V}\cdot\vec{W} = -(a\cosh\lambda\sinh\lambda) + (a\sinh\lambda\cosh\lambda) = 0,$$

which it is.

(b) To prove that this defines a valid coordinate transform from (λ, a) to (t, x), we show that the determinant of the transformation matrix is non-zero.

$$\det\begin{pmatrix}\partial t/\partial\lambda & \partial t/\partial a\\ \partial x/\partial\lambda & \partial x/\partial a\end{pmatrix} = \frac{\partial t}{\partial\lambda}\frac{\partial x}{\partial a} - \frac{\partial t}{\partial a}\frac{\partial x}{\partial\lambda}$$
$$= a\cosh^2\lambda - a\sinh^2\lambda = a$$
$$\neq 0,$$

and so it is indeed a valid coordinate transform.

To plot the curves parameterized by a, we take

$$\begin{aligned}-t^2 + x^2 &= a^2(\cosh^2\lambda - \sinh^2\lambda)\\ &= a^2,\end{aligned}$$

which gives us a family of space-like hyperbola, depending on the chosen value of a.

To plot the curves parameterized by λ, we take

$$\begin{aligned}x &= a\cosh\lambda \implies a = x/\cosh\lambda\\ t &= a\sinh\lambda = x\sinh\lambda/\cosh\lambda = x\tanh\lambda,\end{aligned}$$

which gives us a family of space-like lines, depending on the chosen value of λ.

A plot of these curves is given in Figure 5.2, from which it is clear that only half of the t–x plane is covered. When $|t| = |x|$, then $a = 0$, since $-t^2 + x^2 = a^2$. We already found that the determinant of the coordinate transformation is a, so this would make the determinant 0, making it singular.

(c) Find the metric tensor and Christoffel symbols in (λ, a) coordinates.

First we find the basis vectors:

$$\begin{aligned}\vec{e}_\lambda &= a(\cosh\lambda\vec{e}_t + \sinh\lambda\vec{e}_x),\\ \vec{e}_a &= \sinh\lambda\vec{e}_t + \cosh\lambda\vec{e}_x.\end{aligned}$$

Now we find the components of the metric tensor $\mathbf{g}$ as

$$\begin{aligned}g_{\lambda\lambda} &= a^2(\cosh\lambda\vec{e}_t + \sinh\lambda\vec{e}_x)^2\\ &= a^2(\cosh^2\lambda\eta_{tt} + \sinh^2\lambda\eta_{xx} + 2\sinh\lambda\cosh\lambda\eta_{tx})\\ &= a^2(\sinh^2\lambda - \cosh^2\lambda)\\ &= -a^2\\ g_{aa} &= (\sinh\lambda\vec{e}_t + \cosh\lambda\vec{e}_x)^2\\ &= \sinh^2\lambda\eta_{tt} + \cosh^2\lambda\eta_{xx} + 2\sinh\lambda\cosh\lambda\eta_{tx}\\ &= 1\end{aligned}$$

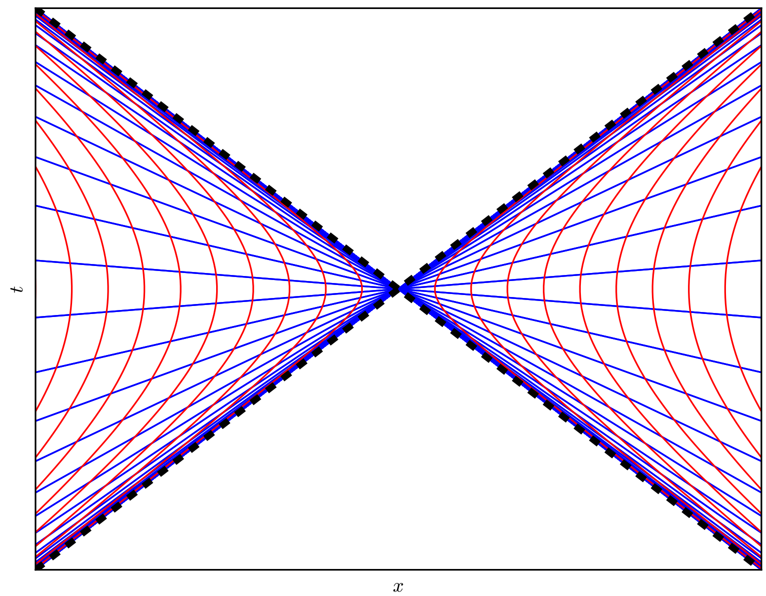

Figure 5.2: Lines of constant λ and a in Problem 21.

$$
\begin{aligned}
g_{\lambda a} = g_{a\lambda} &= a(\cosh\lambda\vec{e}_t + \sinh\lambda\vec{e}_x)(\sinh\lambda\vec{e}_t + \cosh\lambda\vec{e}_x) \\
&= a(\cosh\lambda\sinh\lambda(\eta_{tt} + \eta_{xx}) + 2\sinh\lambda\cosh\lambda\eta_{tx}) \\
&= 0 \\
\mathbf{g} \underset{(\lambda,a)}{\rightarrow} & \begin{pmatrix} -a^2 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

Now for the Christoffel symbols, since we know this is a coordinate basis, we can use

$$
\Gamma^{\gamma}{}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})
$$

$$
\begin{aligned}
\Gamma^{\lambda}{}_{\lambda\lambda} &= \frac{1}{2}g^{\alpha\lambda}(g_{\alpha\lambda,\lambda} + g_{\alpha\lambda,\lambda} - g_{\lambda\lambda,\alpha}) = \frac{1}{2}g^{a\lambda}(-g_{\lambda\lambda,a}) \\
&= 0 \\
\Gamma^{a}{}_{aa} &= \frac{1}{2}g^{\alpha a}(g_{\alpha a,a} + g_{\alpha a,a} - g_{aa,\alpha}) \\
&= 0 \\
\Gamma^{\lambda}{}_{\lambda a} &= \frac{1}{2}g^{\alpha\lambda}(g_{\alpha\lambda,a} + g_{\alpha a,\lambda} - g_{\lambda a,\alpha}) = \frac{1}{2}g^{\lambda\lambda}g_{\lambda\lambda,a} = \frac{1}{2}(-a^{-2})(-2a) \\
&= 1/a \\
\Gamma^{a}{}_{\lambda a} &= \frac{1}{2}g^{\alpha a}(g_{\alpha\lambda,a} + g_{\alpha a,\lambda} - g_{\lambda a,\alpha}) = \frac{1}{2}g^{\lambda a}g_{\lambda\lambda,a}
\end{aligned}
$$

$$
\begin{aligned}
&= 0 \\
\Gamma^{\lambda}{}_{aa} &= \frac{1}{2} g^{\alpha\lambda}(g_{\alpha a,a} + g_{\alpha a,a} - g_{aa,\alpha}) \\
&= 0 \\
\Gamma^{a}{}_{\lambda\lambda} &= \frac{1}{2} g^{\alpha a}(g_{\alpha\lambda,\lambda} + g_{\alpha\lambda,\lambda} - g_{\lambda\lambda,\alpha}) = \frac{1}{2} g^{aa}(-g_{\lambda\lambda,a}) = \frac{1}{2} \cdot 2 \cdot a \\
&= a
\end{aligned}
$$

22

$$
\begin{aligned}
U^{\alpha} \boldsymbol{\nabla}_{\alpha} V^{\beta} = W^{\beta} &\implies U^{\alpha} V^{\gamma}{}_{;\alpha} = W^{\gamma} \\
&\implies g_{\alpha\beta} U^{\alpha} V^{\gamma}{}_{;\alpha} = g_{\gamma\beta} W^{\gamma} \\
&\implies U^{\alpha} V_{\beta;\alpha} = W_{\beta} \\
&\implies U^{\alpha} \boldsymbol{\nabla}_{\alpha} V_{\beta} = W_{\beta}
\end{aligned}
$$