Introduction to General Relativity (ASTP-760) Notes

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Contents

Chapter 1

Special relativity

1.1 Fundamental principles of special relativity (SR) theory

Special relativity can be summarized by two fundamental postulates:

1. The principle of relativity (Galileo), which states that no experiment may measure the absolute velocity of an observer.

2. The universality of the speed of light (Einstein), which states that the speed of light is constant when measured from any inertial reference frame.

1.2 Definition of an inertial observer in SR

When we say "observer", what we really mean is a coordinate system. Thus an inertial observer is a coordinate system that meets the following 3 criteria:

1. The distance between two spatial points P_1 and P_2 is independent of time.

2. Time is synchronized and moves at the same rate at all spatial points.

3. At any constant time, space is Euclidean.

It follows from these criteria that the observer must be **unaccelerated**.

1.3 New units

The speed of light, c, is approximately $3.00 \times 10^8\,\mathrm{ms}^{-1}$ in SI units. However, these units predate relativity, and are very inconvenient. Life becomes easier if we define our units around c, such that $c \equiv 1$.

This can be done by repurposing the meter as a measure of time as well. We thereby define the meter as "the time it takes light to travel 1 meter". Thus the speed of light becomes

$$c = \frac{1\,\mathrm{m}}{1\,\mathrm{m}}.$$

Indeed, it turns out in SR that time is most conveniently measured in distance ($c = 3.00 \times 10^{10}\,\mathrm{cm}$), and in GR mass is as well ($G/c^{-2} = 7.425 \times 10^{-29}\,\mathrm{cm\,g^{-1}}$).

1.4 Spacetime diagrams

1.5 Construction of the coordinates used by another observer

1.6 Invariance of the interval

For two nearby events, we can define the **invariant interval**, defining a 4D Minskowski spacetime:

$$\mathrm{d}s^2 = -(c\,\mathrm{d}t)^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,,$$

or when we set $c \equiv 1$:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,. \qquad \text{(Schutz 1.1)}$$

This notation can be simplified be defining

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \mathrm{d}s^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

When we want to find $\mathrm{d}\bar{s}^2$, we can consider the fact that each of its components, $\mathrm{d}\bar{x}^\alpha$, is a linear combination of the components of $\mathrm{d}s^2$,

$$\mathrm{d}\bar{x}^\alpha = \sum_{\beta=0}^{3} a_{\alpha\beta} x^\beta.$$

Now, when we consider the square of $\mathrm{d}\bar{x}^\alpha$, the cross terms make it a quadratic function. Since the sum of four quadratics (the four $\mathrm{d}\bar{x}^\alpha$'s) is also a quadratic, we can write $\mathrm{d}\bar{s}^2$ as

$$\mathrm{d}\bar{s} = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} M_{\alpha\beta}(\mathrm{d}x^\alpha)(\mathrm{d}x^\beta) \qquad \text{(Schutz 1.2)}$$

If are talking about light, $\mathrm{d}s^2 = 0$, and so we can say

$$\mathrm{d}s^2 = 0 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 \implies \mathrm{d}t = \mathrm{d}r$$

Now by looking at Exercise 8 in Section 1.14, we see that

$$\begin{aligned} \mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &\quad + 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &\quad + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \end{aligned} \qquad \text{(Schutz 1.3)}$$

where

$$M_{0i} = 0 \qquad \text{(Schutz 1.4a)}$$

and

$$M_{ij} = -(M_{00})\delta_{ij}, \qquad \text{(Schutz 1.4b)}$$

where δ_{ij} is the Kronecker delta.

1.7 Invariant hyperbolae

1.8 Particularly important results

1.9 The Lorentz transformation

1.10 The velocity-composition law

1.11 Paradoxes and physical intuition

1.12 Further reading

1.13 Appendix: The twin 'paradox' dissected

Consider two twins, Joe and Ed. Joe goes off in a straight line traveling at a speed of $(24/25)c$ for 7 years, as measured on his clock, then instantaneously reverses and returns at half the speed. Ed remains at home. When they return, what is the difference in ages between Joe and Ed?

$\tau_1 = 7\,\mathrm{yr}$. $t_1 = \tau_1\gamma_1$, where $\gamma_1 = \left[1 - \left(\frac{24}{25}\right)^2\right]^{-1/2}$. So $t_1 = 25\,\mathrm{yr}$.

$t_2 = 2t_1 = 50\,\mathrm{yr}$.

$\tau_2 = t_2\gamma_2^{-1}$, where $\gamma_1 = \left[1 - \left(\frac{12}{25}\right)^2\right]^{-1/2}$. So $\tau_2 = 2\sqrt{481}$yr ≈ 44 yr. Finally, $\tau = \tau_1 + \tau_2 \approx 51$ yr, and $t = t_1 + t_2 = 75$ yr, so Ed ages $t - \tau \approx 24$ years more than Joe.

1.14 Exercises

1 Convert the following to units in which $c = 1$, expressing everything in terms of m and kg.

(Note that $c = 1 \implies 1 \approx 3 \times 10^8\,\mathrm{m\,s^{-1}} \approx (3 \times 10^8)^{-1}\mathrm{m^{-1}\,s}$

(a) 10 J

$$\begin{aligned} 10\,\mathrm{J} = 10\,\mathrm{N\,m} = 10\,\mathrm{kg\,m^2\,s^{-2}} &\approx 10\,\mathrm{kg\,m^2\,s^{-2}} \cdot ((3 \times 10^8)^{-1}\mathrm{m^{-1}\,s})^2 \\ &\approx 10\,\mathrm{kg}(3 \times 10^8)^{-2} = 10\,\mathrm{kg}\left(\frac{1}{9} \times 10^{-16}\right) \approx 1.11 \times 10^{-16}\,\mathrm{kg} \end{aligned}$$

(b) 100 W

$$\begin{aligned} 100\,\mathrm{W} = 100\,\mathrm{kg\,m^2\,s^{-3}} &\approx 100\,\mathrm{kg\,m^2\,s^{-3}} \cdot ((3 \times 10^8)^{-1}\mathrm{m^{-1}\,s})^3 \\ &\approx 100\,\mathrm{kg\,m^{-1}}(3^{-3} \times 10^{-24}) = \frac{100}{27} \times 10^{-24}\mathrm{kg\,m^{-1}} \approx 3.7 \times 10^{-24}\,\mathrm{kg\,m^{-1}} \end{aligned}$$

2 Convert the following from natural units ($c = 1$) to SI units:

(a) A velocity $v = 10^{-2}$.

$$v = 10^{-2} = 10^{-2}c = 10^{-2}3 \times 10^8\,\mathrm{m\,s^{-1}} = 3 \times 10^6\,\mathrm{m\,s^{-1}}$$

(b) Pressure $P = 10^{19}\mathrm{kg\,m^{-3}}$.

$$\begin{aligned} P = 10^{19}\mathrm{kg\,m^{-3}} &\approx 10^{19}\mathrm{kg\,m^{-3}}(3 \times 10^8\,\mathrm{m\,s^{-1}})^2 \\ &\approx 10^{19}\mathrm{kg\,m^{-3}}(9 \times 10^{16}\,\mathrm{m^2\,s^{-2}}) = 9 \times 10^{35}\,\mathrm{N\,m^2} \end{aligned}$$

3 Draw the t and x axes of the spacetime coordinates of an observer $\mathcal{O}$ and then draw:

(a) The world line of $\mathcal{O}$'s clock at $x = 1$ m.

4 Write out all the terms of the following sums, substituting the coordinate names (t, x, y, z) for (x^0, x^1, x^2, x^3):

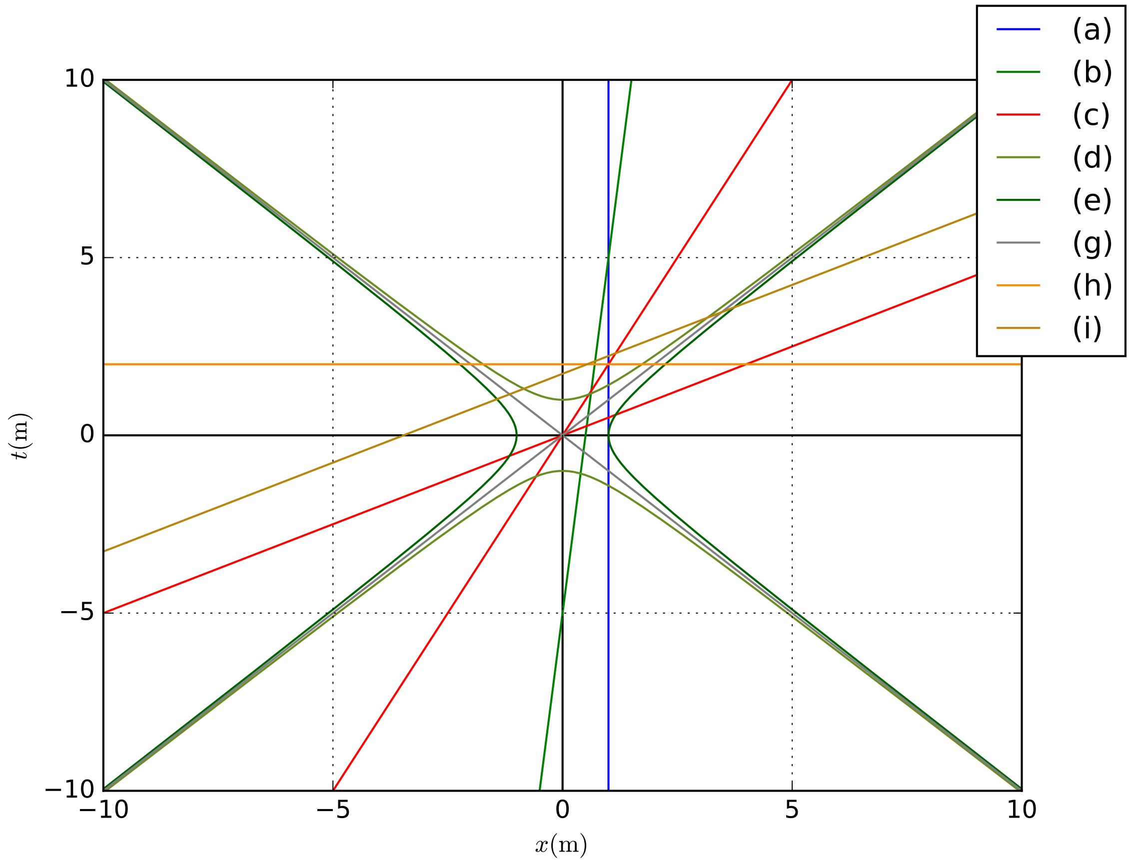
(a) $\sum_{\alpha=0}^{3} V_\alpha\,\mathrm{d}x^\alpha = V_0\,\mathrm{d}t + V_1\,\mathrm{d}x + V_2\,\mathrm{d}y + V_3\,\mathrm{d}z$.

(b) $\sum_{i=1}^{3}(\mathrm{d}x^i)^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2$.

5

(a) Use the spacetime diagram of an observer $\mathcal{O}$ to describe the following experiment performed by $\mathcal{O}$. Two bursts of particles of speed $v = 0.5$ are emitted from $x = 0$ at $t = -2$ m, one traveling in the $+x$ direction and the other in the $-x$ direction. These encounter detectors located at $x = \pm 2$ m. After a delay of 0.5 m of time, the detectors send signals back to $x = 0$ at speed $v = 0.75$.

See figure below



Exercise 3

(b) The signals arrive back at $x = 0$ at the same event. (Make sure your spacetime diagram shows this!) From this the experimenter concludes that the particle detectors did indeed send out their signals simultaneously, since he knows they are equal distances from $x = 0$. Explain why this conclusion is valid.

Assuming he knows the signals traveled with equal speeds, and the detectors are an equal distance away, then they must have been emitted simultaneously, in order for them to arrive at $x = 0$ simultaneously.

(c) A second observer $\bar{\mathcal{O}}$ moves with speed $v = 0.75$ in the $-x$ direction relative to $\mathcal{O}$. Draw the spacetime diagram of $\bar{\mathcal{O}}$ and in it depict the experiment performed by $\mathcal{O}$. Does $\bar{\mathcal{O}}$ conclude that particle detectors sent out their signals simultaneously? If not, which signal was sent first.

See the diagram below. On it, I have drawn lines $\bar{t}_{\text{left}}$ and $\bar{t}_{\text{right}}$ (note that they are parallel to the $\bar{x}$ axis). As you can see from the plot, the left emission occurs *before* the right emission.

(d)

Using $\mathcal{O}$, the distance is

$$\Delta s^2 = \Delta x^2 = 16\,\text{m}^2.$$

Using $\bar{\mathcal{O}}$, we first need to find $\bar{x}_{\{a,b\}}$ and $\bar{t}_{\{a,b\}}$. We use the Lorentz transformation to do this.

$$\bar{t} = \gamma(t - vx)$$

$$\bar{x} = \gamma(x - vt)$$

Using this, we find

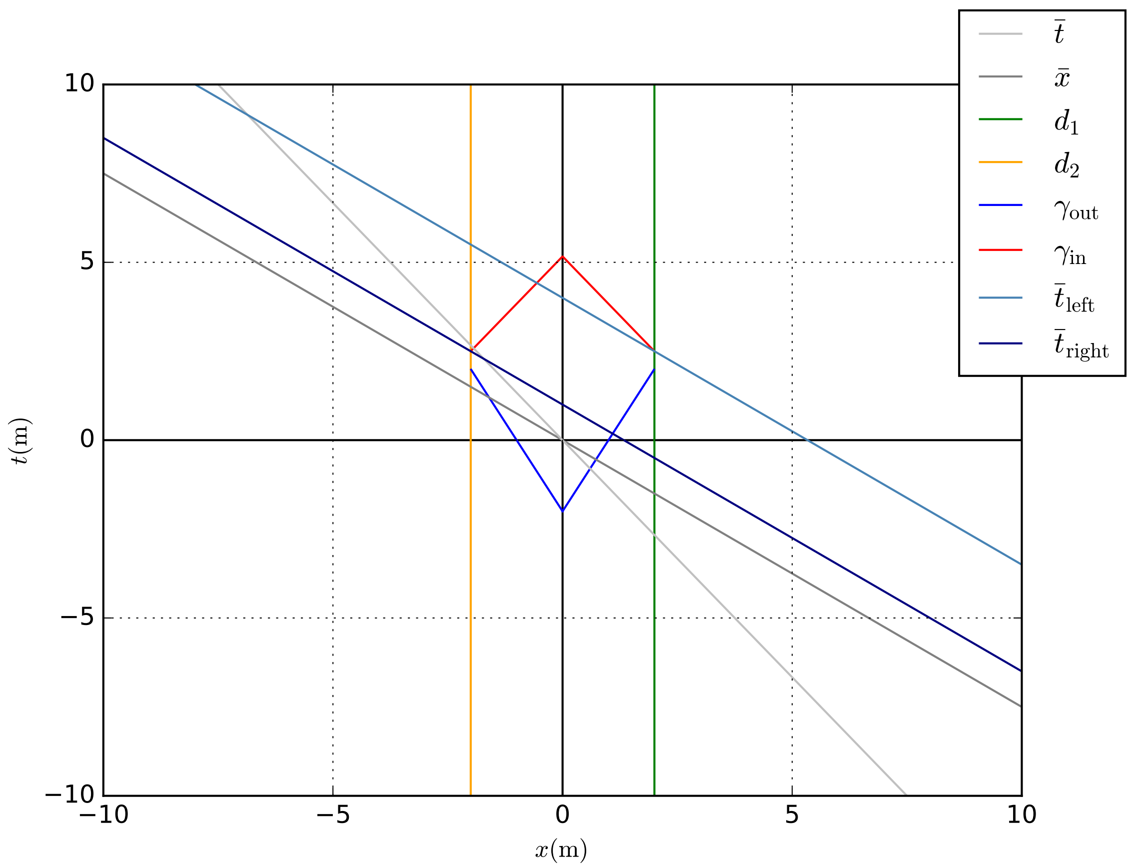
$$\bar{t}_a = \frac{16\sqrt{7}}{7} \qquad \bar{t}_b = \frac{4\sqrt{7}}{7}$$

$$\bar{x}_a = \frac{-31\sqrt{7}}{14} \qquad \bar{x}_b = \frac{\sqrt{7}}{14}$$

This gives us a distance of

$$\Delta\bar{s}^2 = -(\Delta\bar{t})^2 + (\Delta\bar{x})^2 = 16\,\mathrm{m}^2,$$

which is of course what we expect.



Exercise 5

6 Show that Equation (Schutz 1.2) contains only $M_{\alpha\beta} + M_{\beta\alpha}$ when $\alpha \neq \beta$, not $M_{\alpha\beta}$ and $M_{\beta\alpha}$ independently. Argue that this enables us to set $M_{\alpha\beta} = M_{\beta\alpha}$ without loss of generality.

When we expand the summation in (Schutz 1.2), there is no point where

$$d\bar{s}^2 = \ldots + M_{\alpha\alpha}(dx^\alpha)^2 + M_{\alpha\alpha}(dx^\alpha)^2 + \ldots$$

occurs, because a double summation only contains $M_{\alpha\alpha}$ once. If it did, we could absorb the two $M_{\alpha\beta}$ terms into a single one. Therefore we can assert the first point.

Now we consider the second point. If we expand the summation, assuming now that an $M_{\alpha\beta}$ and $M_{\beta\alpha}$ term only occur when $\alpha \neq \beta$, then we see

$$\begin{aligned} d\bar{s}^2 &= \ldots + M_{\alpha\beta}(dx^\alpha)(dx^\beta) + M_{\beta\alpha}(dx^\beta)(dx^\alpha) + \ldots \\ &= \ldots + (M_{\alpha\beta} + M_{\beta\alpha})\Big[(dx^\alpha)(dx^\beta)\Big] + \ldots \\ &= \ldots + \mathbf{X}\Big[(dx^\alpha)(dx^\beta)\Big] + \ldots. \end{aligned}$$

Now, what really matters in this summation is the value of $\mathbf{X} = M_{\alpha\beta} + M_{\beta\alpha}$, not the individual values of $M_{\alpha\beta}$ and $M_{\beta\alpha}$. Therefore we can *choose*, without loss of generality, $M_{\alpha\beta} = M_{\beta\alpha} = \mathbf{X}/2$, thereby asserting the second point.

7 In the discussion leading up to Equation (Schutz 1.2), assume that the coordinates of $\bar{\mathcal{O}}$ are given as the following linear combinations of those $\mathcal{O}$:

$$\begin{aligned} \bar{t} &= \alpha t + \beta x, \\ \bar{x} &= \mu t + \nu x, \\ \bar{y} &= a y, \\ \bar{z} &= b z, \end{aligned}$$

where α, β, μ, ν, a, and b may be functions of the velocity $\vec{v}$ of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$, but they do not depend on the coordinates. Find the values of $M_{\alpha\beta}$ of Equation (Schutz 1.2).

$$\begin{aligned} d\bar{s}^2 &= -(d\bar{t})^2 + (d\bar{x})^2 + (d\bar{y})^2 + (d\bar{z})^2 \\ &= -(\alpha\, dt + \beta\, dx)^2 + (\mu\, dt + \nu\, dx)^2 + (a\, dy)^2 + (b\, dz)^2 \\ &= -\alpha^2\, dt^2 - \alpha\beta\, dt\, dx - \beta^2\, dx^2 + \mu^2\, dt^2 + \mu\nu\, dt\, dx + \nu^2\, dx^2 + a^2\, dy^2 + b^2\, dz^2 \\ &= (\mu^2 - \alpha^2)\, dt^2 + (\mu\nu - \alpha\beta)\, dt\, dx + (\nu^2 - \beta^2)\, dx^2 + a^2\, dy^2 + b^2\, dz^2 \\ M_{00} &= \mu^2 - \alpha^2 \\ M_{01} &= M_{10} = \frac{\mu\nu - \alpha\beta}{2} \\ M_{11} &= \nu^2 - \beta^2 \\ M_{22} &= a^2 \\ M_{33} &= b^2, \end{aligned}$$

and all other $M_{\alpha\beta} = 0$.

8

(a) Derive Equation (Schutz 1.3) from (Schutz 1.2) for general $M_{\alpha\beta}$.

Equation (Schutz 1.3) is just an expansion of the summation in (Schutz 1.2).

We start by taking out the $\mathrm{d}t^2$ term, which corresponds to $\alpha = \beta = 0$, which gives us

$$\mathrm{d}\bar{s}^2 = M_{00}(\mathrm{d}t)^2 + \dots,$$

now we use the equivalence of $\mathrm{d}t$ and $\mathrm{d}r$ to make the substitution

$$\mathrm{d}\bar{s}^2 = M_{00}(\mathrm{d}r)^2 + \dots.$$

For the middle terms, we use the fact that $M_{\alpha\beta} = M_{\beta\alpha}$, and look at only the terms where *one* of α and β is zero. The symmetry means we can write $M_{0i} = M_{i0}$, and pull out a 2 because there are twice as many terms, giving us

$$\begin{aligned}\mathrm{d}\bar{s}^2 = {} & M_{00}(\mathrm{d}r)^2 \\ & + 2\left(\sum_{i=1}^{3} M_{0i}(\mathrm{d}x^i)(\mathrm{d}t)\right) \\ & + \dots.\end{aligned}$$

Now we use the equivalence of $\mathrm{d}t$ and $\mathrm{d}r$ once again, and pull the term out of the sum, giving us

$$\begin{aligned}\mathrm{d}\bar{s}^2 = {} & M_{00}(\mathrm{d}r)^2 \\ & + 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ & + \dots.\end{aligned}$$

Finally, we simply include the terms which have not yet been accounted for, which are all the *spacial-only* terms, which arrives us back at Equation (Schutz 1.3):

$$\begin{aligned}\mathrm{d}\bar{s}^2 = {} & M_{00}(\mathrm{d}r)^2 \\ & + 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ & + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,.\end{aligned}$$

(b) Since $\mathrm{d}\bar{s}^2 = 0$ in Equation (Schutz 1.3), for *any* $\mathrm{d}x^i$, replace $\mathrm{d}x^i$ with $-\mathrm{d}x^i$, and subtract that result from the original equation. This will establish that $M_{0i} = 0$.

$$\begin{aligned} \mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &- 2\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \end{aligned}$$

$$\begin{aligned} \mathrm{d}\bar{s}^2 - \mathrm{d}\bar{s}^2 = 0 &= 0\,\cancel{M_{00}(\mathrm{d}r)^2} \\ &+ 4\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\mathrm{d}r \\ &+ 0\,\cancel{\sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}\,. \end{aligned}$$

$$0 = \cancel{4}\left(\sum_{i=1}^{3} M_{0i}\,\mathrm{d}x^i\right)\cancel{\mathrm{d}r}$$

Now there are two possibilities. In one case, $\mathrm{d}x^i \equiv 0$, but that is a trivial solution and in general is not true. The other case is that $M_{0i} \equiv 0$, which means we can simplify Equation (Schutz 1.3) to

$$\begin{aligned} \mathrm{d}\bar{s}^2 &= M_{00}(\mathrm{d}r)^2 \\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \end{aligned}$$

(c) Use the result of part (b) with $\mathrm{d}\bar{s}^2 = 0$ to establish Equation (Schutz 1.4b).

$$\begin{aligned} \mathrm{d}\bar{s}^2 = 0 &= M_{00}(\mathrm{d}r)^2 + \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \\ \implies -M_{00}(\mathrm{d}r)^2 &= \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \end{aligned}$$

now if we expand $(\mathrm{d}r)^2$, we see that there can only be non-zero M_{ij} when $i = j$, and so

$$\begin{aligned} -M_{00}\Big((\mathrm{d}x^2) + (\mathrm{d}y^2) + (\mathrm{d}z^2)\Big) &= \sum_{i=1}^{3} M_{ii}(\mathrm{d}x^i)^2 \\ \implies -(M_{00})\delta_{ij} &= M_{ij}, \end{aligned}$$

which is simply Equation (Schutz 1.4b).

9 Explain why the line $\mathcal{PL}$ in Figure 1.7 is drawn in the manner described in the text.

10 For the pairs of events whose coordinates (t, x, y, z) in some frame are given below, classify their separations as timelike, spacelike, or null.

(a) $(0,0,0,0)$ and $(-1,1,0,0)$:

$$\mathrm{d}s^2 = -(0+1)^2 + (0-1)^2 + (0-0)^2 + (0-0)^2 = -1+1+0+0 = 0 \implies \text{null}$$

(b) $(1,1,-1,0)$ and $(-1,1,0,2)$:

$$\mathrm{d}s^2 = -(1+1)^2 + (1-1)^2 + (-1-0)^2 + (0-2)^2 = -4+0+1+4 = 1 \implies \text{spacelike}$$

(c) $(6,0,1,0)$ and $(5,0,1,0)$:

$$\mathrm{d}s^2 = -(6-5)^2 + (0-0)^2 + (1-1)^2 + (0-0)^2 = -1+0+0+0 = -1 \implies \text{timelike}$$

(d) $(-1,1,-1,1)$ and $(4,1,-1,6)$:

$$\mathrm{d}s^2 = -(-1-4)^2 + (1-1)^2 + (-1+1)^2 + (1-6)^2 = -25+0+0+25 = 0 \implies \text{null}$$

11 Show that the hyperbolae $-t^2 + x^2 = a^2$ and $-t^2 + x^2 = -b^2$ are asymptotic to the lines $t = \pm x$, regardless of a and b.

We will generalize a and $-b$ with a new constant, $\alpha \in \mathbb{R}$, and so we have: $-t^2 + x^2 = \alpha^2$. Now if we solve for t, we get $t = \pm\sqrt{x^2 - \alpha^2}$.

Now take the limit of t as $x \to \infty$ (or $-\infty$, they are equivalent since x is real and squared), which gives us:

$$\lim_{x\to\infty} t = \lim_{x\to\infty} \pm\sqrt{x^2-\alpha^2} = \pm\sqrt{x^2} = \pm x.$$

Note that we dropped the α^2 term in the limit, as it was being subtracted from a number approaching infinity, and was therefore negligible.

12

(a) Use the fact that the tangent to the hyperbola $\mathcal{DB}$ in Figure 1.14 is the line of simultaneity for $\bar{\mathcal{O}}$ to show that the time interval $\mathcal{AE}$ is shorter than the time recorded on $\bar{\mathcal{O}}$'s clock as it moved from $\mathcal{A}$ to $\mathcal{B}$.

If we look at the figure, we see that $\mathcal{AD}$ and $\mathcal{AB}$ lie along the same hyperbola. This means that when $\mathcal{O}$ measures $\mathrm{d}t = \mathcal{AD}$, and $\bar{\mathcal{O}}$ measures $\mathrm{d}\bar{t} = \mathcal{AB}$, the two measurements are the same. Since $\mathrm{d}t = \mathcal{AE}$ is clearly shorter than $\mathrm{d}t = \mathcal{AD}$, then $\mathrm{d}t = \mathcal{AD} < \mathrm{d}\bar{t} = \mathcal{AB}$.

(b) Calculate that

$$(\mathrm{d}s^2)_{\mathcal{AC}} = (1-v^2)(\mathrm{d}s^2)_{\mathcal{AB}}$$

$$\begin{aligned}
(\mathrm{d}s^2)_{\mathcal{AC}} &= -(\mathrm{d}t)^2_{\mathcal{AC}} \\
(\mathrm{d}s^2)_{\mathcal{AB}} &= (\mathrm{d}\bar{s}^2)_{\mathcal{AB}} \\
&= -(\mathrm{d}\bar{t})^2_{\mathcal{AB}} \\
&= -(\gamma(\mathrm{d}t - v\,\mathrm{d}x))^2 = -(\gamma(\mathrm{d}t - v\cdot 0))^2 = -(\gamma\,\mathrm{d}t)^2 = \gamma^2[-(\mathrm{d}t)^2] \\
&= \gamma^2(\mathrm{d}s^2)_{\mathcal{AC}} = \frac{(\mathrm{d}s^2)_{\mathcal{AC}}}{1-v^2}
\end{aligned}$$

$$\implies (\mathrm{d}s^2)_{\mathcal{AC}} = (1-v^2)(\mathrm{d}s^2)_{\mathcal{AB}}$$

13 The Half-life of the elementary particle called the π-meson (or pion) is $2.5 \times 10^{-8}\,\mathrm{s}$ when the pion is at rest relative to the observer measuring its decay time. Show, by the principle of relativity, that pions moving at speed $v = 0.999$ must have a half-life of $5.6 \times 10^{-7}\,\mathrm{s}$, as measured by an observer at rest.

$$\mathrm{d}t = \gamma\,\mathrm{d}\bar{t} = \frac{2.5 \times 10^{-8}\,\mathrm{s}}{\sqrt{1-0.999^2}} \approx 5.59 \times 10^{-7}\,\mathrm{s}$$

14 Suppose the velocity $\mathbf{v}$ of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$ is small, $|\mathbf{v}| \ll 1$. Show that the time dilation, Lorentz contraction, and velocity-addition formulae can be approximated by respectively:
(a) $\mathrm{d}t \approx (1 + \frac{1}{2}v^2)\,\mathrm{d}\bar{t}$

$$\begin{aligned}
\gamma = (1-v^2)^{-1/2} &= \sum_{k=0}^{\infty}\binom{-1/2}{k}x^k = 1 + (-1/2)(-v^2) + \frac{(-1/2)(-1/2-1)}{2!}(-v^2)^2 + \ldots \approx 1 + \frac{1}{2}v^2 \\
\mathrm{d}t = \gamma\,\mathrm{d}\bar{t} &\approx \left(1 + \frac{1}{2}v^2\right)\mathrm{d}\bar{t}
\end{aligned}$$

(b) $\mathrm{d}x \approx (1 - \frac{1}{2}v^2)\,\mathrm{d}\bar{x}$

$$\begin{aligned}
\gamma^{-1} = (1-v^2)^{1/2} &= \sum_{k=0}^{\infty}\binom{1/2}{k}x^k = 1 + (1/2)(-v^2) + \frac{(1/2)(1/2-1)}{2!}(-v^2)^2 + \ldots \approx 1 - \frac{1}{2}v^2 \\
\mathrm{d}x = \gamma^{-1}\,\mathrm{d}\bar{x} &\approx \left(1 - \frac{1}{2}v^2\right)\mathrm{d}\bar{x}
\end{aligned}$$

(c) $W' \approx W + v - Wv(W+v)$ (with $|W| \ll 1$ as well)

$$\begin{aligned}
W' &= \frac{W+v}{1+Wv} = (W+v)(1+Wv)^{-1} \\
(1+Wv)^{-1} &= \sum_{k=0}^{\infty}\binom{-1}{k}(Wv)^k = 1 - Wv + \frac{1}{2}\cdot 1(1+1)(Wv)^2 - \frac{1}{6}\cdot 1(1+1)(1+2)(Wv)^3 + \ldots \\
&\approx 1 - Wv + (Wv)^2 \\
W' &\approx (W+v)(1 - Wv + (Wv)^2) = W + v - Wv(W+v) + (Wv)^2(W+v) \\
&\approx W + v - Wv(W+v)
\end{aligned}$$

What are the relative errors in these approximations when $|\mathbf{v}| = W = 0.1$?

TODO

15 Suppose that the velocity $\mathbf{v}$ of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$ is nearly that of light, $|\mathbf{v}| = 1 - \varepsilon$, $0 < \varepsilon \ll 1$. Show that the same formulae of Exercise 14 become
(a) $\mathrm{d}t \approx \mathrm{d}\bar{t}/\sqrt{2\varepsilon}$

$$
\begin{aligned}
v = 1 - \varepsilon \implies v^2 &= (1-\varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2 \\
\implies 1 - v^2 = 1 - (1 - 2\varepsilon + \varepsilon^2) &= 2\varepsilon - \varepsilon^2 = 2\varepsilon\left(1 - \frac{\varepsilon}{2}\right) \\
\gamma = (1-v^2)^{-1/2} &= \left(2\varepsilon\left(1 - \frac{\varepsilon}{2}\right)\right)^{-1/2} = \frac{1}{\sqrt{2\varepsilon}}\left(1 - \frac{\varepsilon}{2}\right)^{-1/2} \approx \frac{1}{\sqrt{2\varepsilon}} \\
\mathrm{d}t = \gamma\, \mathrm{d}\bar{t} &\approx \frac{\mathrm{d}\bar{t}}{\sqrt{2\varepsilon}}
\end{aligned}
$$

(b) $\mathrm{d}x \approx \mathrm{d}\bar{x}\,\sqrt{2\varepsilon}$

$$
\begin{aligned}
v = 1 - \varepsilon \implies v^2 &= (1-\varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2 \\
\implies 1 - v^2 = 1 - (1 - 2\varepsilon + \varepsilon^2) &= 2\varepsilon - \varepsilon^2 = 2\varepsilon\left(1 - \frac{\varepsilon}{2}\right) \\
\gamma^{-1/2} = (1-v^2)^{1/2} &= \left(2\varepsilon\left(1 - \frac{\varepsilon}{2}\right)\right)^{1/2} = \sqrt{2\varepsilon}\left(1 - \frac{\varepsilon}{2}\right)^{1/2} \approx \sqrt{2\varepsilon} \\
\mathrm{d}x = \gamma^{-1}\, \mathrm{d}\bar{x} &\approx \mathrm{d}\bar{t}\,\sqrt{2\varepsilon}
\end{aligned}
$$

(c) $W' \approx 1 - \varepsilon(1-W)/(1+W)$

TODO

What are the relative errors on these approximations when $\varepsilon = 0.1$ and $W = 0.9$?

TODO

16 Use the Lorentz transformation, Equation 1.12, to derive (a) the time dilation, and (b) the Lorentz contraction formulae. Do this by identifying pairs of events where the separations (in time or space) are to be compared, and then using the Lorentz transformation to accomplish the algebra that the invariant hyperb b olae had been used for in the text.

(a) To derive the time dilation formula, we choose two events that occur at $x = c$, and times t_1 and t_2. Thus, from $\mathcal{O}$'s frame, the time elapsed between these two events is $\Delta t = t_2 - t_1$, and the distance between them is $\Delta x = 0$. Another observer, $\bar{\mathcal{O}}$, moves with some velocity v relative to $\mathcal{O}$. As it passes through the lines $t = t_1$ and $t = t_2$, its clock moves forward by a time $\Delta\tau = \bar{t}_2 - \bar{t}_1$. We now use the Lorentz transformation to write $\Delta\tau$ in terms of $\mathcal{O}$'s coordinates.

$$
\begin{aligned}
\Delta\tau = \bar{t}_2 - \bar{t}_1 &= \gamma\big[(t_2 - vx_2) - (t_1 - vx_1)\big] = \gamma\big[(t_2 - t_1) + (vx_1 - vx_2)\big] \\
&= \gamma[\Delta t + v\Delta x] = \gamma[\Delta t + v \cdot 0] \\
&= \gamma\Delta t
\end{aligned}
$$

and thus we have arrived at the formula for time dilation.

(b) To derive the Lorentz contraction formula, we take a slightly different approach. In the $\mathcal{O}$ frame, a stick lies parallel to x, such that its length $\ell = x_2 - x_1$. In this frame, the world lines of the two ends of the stick form vertical lines. Another observer, $\bar{\mathcal{O}}$, moves with a velocity v, relative to $\mathcal{O}$. Two events, $\mathcal{A}$ and $\mathcal{B}$ occur on either end of the stick, such that $\bar{\mathcal{O}}$ observes the two events to be simultaneous. Thus, from the $\bar{\mathcal{O}}$ frame, the events are located a distance $\Delta\bar{x} = \bar{\ell}$ apart, and $\Delta\bar{t} = 0$. However, from the $\mathcal{O}$ frame, the events occur a distance $\Delta x = \ell$ apart, and a time separation $\Delta t \neq 0$.

$$\ell = x_2 - x_1 = \gamma\big[(\bar{x}_2 - v\bar{t}_2) - (\bar{x}_1 - v\bar{t}_1)\big] = \gamma\big[(\bar{x}_2 - \bar{x}_1) + v(\bar{t}_1 - \bar{t}_2)\big] = \gamma\bar{\ell}$$
$$\implies \bar{\ell} = \frac{\ell}{\gamma}$$

17 A lightweight pole, 20 m long, lies on the ground next to a barn 15 m long. An Olympic athlete picks up the pole, carries it far away, and runs with it toward the end of the barn at a speed 0.8. His friend remains at rest, standing by the door of the barn. Attempt all parts of this question, even if you can't answer some.
(a) How long does the friend measure the pole to be, as it approaches the barn?
We use the Lorentz contraction equation to find the length the friend measures.

$$\bar{\ell} = \ell/\gamma = \ell\sqrt{1 - v^2} = 20\,\text{m}\sqrt{1 - 0.8^2} = 12\,\text{m}$$

(b) The barn door is initially open and, immediately after the runner and pole are entirely inside the barn, the friend shuts the door. How long after the door is shut does the front of the pole hit the other end of the barn, as measured by the friend? Compute the interval between the events of shutting the door and hitting the wall. Is it spacelike, timelike, or null?
From the runner's point of view, we must consider the length contraction of the barn
(c) In the reference frame of the runner, what is the length of the barn and the pole?
(d) Does the runner believe that the pole is entirely inside the barn when its front hits the end of the barn? Can you explain why?
(e) After the collision, the pole and runner come to rest relative to the barn. From the friend's point of view, the 20 m pole is now inside a 15 m barn, since the barn door was shut before the pole stopped. How is this possible? Alternatively, from the runner's point of view, the collision should have stopped the pole *before* the door closed, so the door could not be closed at all. Was or was not the door closed with the pole inside?
(f) Draw a spacetime diagram from the friend's point of view and use it to illustrate and justify all your conclusions.
18
(a) The Einstein velocity-addition law, Equation 1.13, has a simpler form if we introduce the concept of the *velocity parameter* u, defined by the equation

$$v = \tanh u.$$

Notice that for $-\infty < u < \infty$, the velocity is confined to the acceptable limits $-1 < v < 1$. Show that if

$$v = \tanh u$$

and

$$w = \tanh U,$$

then Equation 1.13 implies

$$w' = \tanh(u + U).$$

This means that velocity parameters add linearly.

There exists an identity:

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$$

If we simply use $x = u$ and $y = U$, then we arrive at

$$\tanh(u + U) = \frac{\tanh(u) + \tanh(U)}{1 + \tanh(u)\tanh(U)} = w'$$

(b) Use this to solve the following problem. A star measures a second star to be moving away at speed $v = 0.9$. The second star measures a third to be receding in the same direction at 0.9. Similarly, the third measures a fourth, and so on, up to some large number N of stars. What is the velocity of the Nth star relative to the first? Give an exact answer and an approximation useful for large N.

Let w^N be the velocity of the Nth star relative to the original star, which we will call star 0. We will use an induction proof to find an expression for w^N. The base case is trivial, $w^0 = 0$, as the star does not move relative to itself. For the next case, $w^1 = v$, we still aren't really doing velocity addition, so we will skip to the w^2 case, where things get interesting, though we will later show that the general expression holds for w^0 and w^1.

For w^2, we simply use the Einstein velocity-addition law:

$$w^2 = \tanh(u + U) = \tanh\Big(\tanh^{-1} v + \tanh^{-1} w^1\Big) = \tanh\Big(2\tanh^{-1} v\Big).$$

Now I will prove that this is one instance of a general expression, that $w^N = \tanh\Big(N\tanh^{-1} v\Big)$.

$$\begin{aligned}
w^N &= \tanh\Big(N\tanh^{-1} v\Big) \\
\implies \tanh^{-1} w^N &= N\tanh^{-1} v \\
\implies \tanh^{-1} w^N + \tanh^{-1} v &= N\tanh^{-1} v + \tanh^{-1} v \\
\implies \tanh^{-1} w^N + \tanh^{-1} v &= (N+1)\tanh^{-1} v \\
\implies \tanh\Big(\tanh^{-1} w^N + \tanh^{-1} v\Big) &= \tanh\Big((N+1)\tanh^{-1} v\Big) \\
\implies w^{N+1} &= \tanh\Big((N+1)\tanh^{-1} v\Big).
\end{aligned}$$

If you can believe the last step, then this is proof that it works for all N. The last step is saying that, if we have a star N, moving away from star 0 at a speed w^N, and another star $N+1$, moving away from star N at a speed v, then star $N+1$ as observed from star 0 is given by the Einstein velocity-addition law, meaning we can rewrite that expression as w^{N+1}.

Now I'd like to go back and show that this works for $N = 0$ and $N = 1$. For $N = 1$, we get

$$w^1 = \tanh\Big(1 \tanh^{-1} v\Big) = v,$$

which is what we would expect, and for $N = 0$, we get

$$w^0 = \tanh\Big(0 \tanh^{-1} v\Big) = 0,$$

which we also expect. So the general expression,

$$w^N = \tanh\Big(N \tanh^{-1} v\Big),$$

holds true for all non-negative integers N. We can also write this more elegantly as

$$w^N = \tanh(Nu).$$

Now we want to consider the behaviour at large N. We first write tanh in its exponential form, as

$$w^N = \frac{1 - \exp(-2Nu)}{1 + \exp(-2Nu)}.$$

When N is very large, then the exponential in the bottom term goes to zero, allowing us to rewrite it as

$$w^N \approx 1 - \exp(-2Nu).$$

We can go a step further. Since $v = 0.9$, $u \approx 1.47$, which we can neglect for large N, and so we finally arrive at

$$w^N \approx 1 - \exp(-2N).$$

19

(a) Using the velocity parameter (u) introduced in Exercise 18, show that the Lorentz transformation equations, Equation 1.12, can be put in the form

$$\bar{t} = t\cosh u - x\sinh u \qquad \bar{y} = y$$
$$\bar{x} = -t\sinh u + x\cosh u \qquad \bar{z} = z$$

We start by putting γ in terms of u.

$$\gamma = (1 - v^2)^{-1/2} = (1 - \tanh^2 u)^{-1/2} = \frac{1}{\operatorname{sech} u} = \cosh u.$$

Now we can substitute this into the Lorentz transformation equations

$$\begin{aligned}\bar{t} &= \gamma(t - vx) = \cosh u(t - x\tanh u) = t\cosh u - x\sinh u \\ \bar{x} &= \gamma(x - vt) = \cosh u(x - t\tanh u) = x\cosh u - t\sinh u\end{aligned}$$

(b) Use the identity $\cosh^2 u - \sinh^2 u = 1$ to demonstrate the invariance of the interval from these equations.

$$\begin{aligned}\mathrm{d}s^2 &= -\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ \mathrm{d}\bar{s}^2 &= -(\mathrm{d}t\cosh u - \mathrm{d}x\sinh u)^2 + (\mathrm{d}x\cosh u - \mathrm{d}t\sinh u)^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -\Big(\mathrm{d}t^2\cosh^2 u - \cancel{\mathrm{d}x\,\mathrm{d}t\sinh u\cosh u} + \mathrm{d}x^2\sinh^2 u\Big) \\ &+ \Big(\mathrm{d}x^2\cosh^2 u - \cancel{\mathrm{d}t\,\mathrm{d}x\sinh u\cosh u} + \mathrm{d}t^2\sinh^2 u\Big) + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -\,\mathrm{d}t^2\cancel{\Big(\cosh^2 u - \sinh^2 u\Big)} + \mathrm{d}x^2\cancel{\Big(\cosh^2 u - \sinh^2 u\Big)} + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= \mathrm{d}s^2\end{aligned}$$

(c) Draw as many parallels as you can between the geometry of spacetime and ordinary two-dimensional Euclidean geometry, where the coordinate transformation analogous to the Lorentz transformation is

$$\begin{aligned}\bar{x} &= +x\cos\theta + y\sin\theta, \\ \bar{y} &= -x\sin\theta + y\cos\theta.\end{aligned}$$

What is the analog of the interval? Of the invariant hyperbolae?

The analog of the interval would be

$$\begin{aligned}\mathrm{d}\bar{r}^2 = \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2 &= (\mathrm{d}x\cos\theta + \mathrm{d}y\sin\theta)^2 + (\mathrm{d}y\cos\theta - \mathrm{d}x\sin\theta)^2 + \\ &= \mathrm{d}x^2\cos^2\theta + \cancel{2\,\mathrm{d}x\,\mathrm{d}y\sin\theta\cos\theta} + \mathrm{d}y^2\sin^2\theta \\ &+ \mathrm{d}y^2\cos^2\theta - \cancel{2\,\mathrm{d}x\,\mathrm{d}y\sin\theta\cos\theta} + \mathrm{d}x^2\sin^2\theta \\ &= \mathrm{d}x^2\left(\sin^2\theta + \cos^2\theta\right) + \mathrm{d}y^2\left(\sin^2\theta + \cos^2\theta\right) \\ &= \mathrm{d}x^2 + \mathrm{d}y^2\end{aligned}$$

The analog of the invariant hyperbola would be the invariant circle, as $\bar{x}$ and $\bar{y}$ are both equations of a circle.

20 Write the Lorentz transformation equations in matrix form.

$$\begin{aligned}\bar{t} &= \gamma(t - vx) \\ \bar{x} &= \gamma(x - vt) \\ \bar{y} &= y \\ \bar{z} &= z\end{aligned}$$

$$\begin{aligned}\bar{t} &= \gamma t - \gamma vx + 0y + 0z \\ \bar{x} &= -\gamma vt + \gamma x + 0y + 0z \\ \bar{y} &= y \\ \bar{z} &= z\end{aligned}$$

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

21

(a) Show that if the two events are timelike separated, there is a Lorentz frame in which they occur at the same point, i.e. at the same spatial coordinate values.

If the two events are timelike separated, then it must be possible to have an object with a worldline which crosses the two points, as it is inside the light cone. If such an object exists, then we can draw a Lorentz frame for it, so its time axis, $\bar{t}$ is that line, meaning $\bar{x} = 0$ for both events.

(b) Similarly, if the two events are spacelike separated, there is a Lorentz frame in which they occur simultaneously.

If the two events are spacelike separated, then it must be possible to draw a coordinate frame where $\bar{x}$ has slope v in $\mathcal{O}$'s frame. This means that $\bar{t} = 0$ for both events, and so they are simultaneous.

Chapter 2

Vector analysis in special relativity

2.1 Definition of a vector

2.2 Vector algebra

2.3 The four-velocity

An object's four velocity, denoted $\vec{U}$, is the vector tangent to its world line, with unit length. This means it extends one unit in time, and zero in space, so it is timelike.

For an *accelerated* particle (which we have not considered up to now), we may not be able to define an inertial frame, but we *can* define a **momentarily comoving reference frame** (MCRF) which, as the name suggests, moves with the same velocity as the observer for an infinitesimal period of time. We can therefore construct a continuous sequence of MCRFs for any object. If an object has MCRF $\mathcal{O}$, then its four-velocity is *defined* to be the basis vector $\vec{e}_0$.

2.4 The four-momentum

Analogous to the three-momentum, we define the four-momentum to be

$$\vec{p} = m\vec{U}. \qquad \text{(Schutz 2.19)}$$

It has components

$$\vec{p} \underset{\mathcal{O}}{\rightarrow} (E, p^1, p^2, p^3). \qquad \text{(Schutz 2.20)}$$

Calling p^0 "E" is no accident, it is in fact the energy. There is an interesting consequence to this: since vectors are invariant with respect to reference frame, but vector components are not, this means that the four-momentum does not change in different reference frames, but the energy *does*. One example would be the doppler effect, which causes the color (or energy) of a photon to shift depending on the radial velocity of the source and observer.

2.5 Scalar product

$$\vec{A} \cdot \vec{B} = -(A^0 B^0) + (A^1 B^1) + (A^2 B^2) + (A^3 B^3)$$

2.6 Applications

2.7 Photons

$\vec{x} \cdot \vec{x} = 0$, so we cannot define $\vec{U}$ for photons. We can, however, define $\vec{p}$. Since $\vec{p} \cdot \vec{p} = -m^2$, and photons are massless, we have $\vec{p} \cdot \vec{p} = 0$.

2.8 Further reading

2.9 Exercises

2 Identify the free and dummy indices in the following equations, and write equivalent expressions with different indices. Also, write how many equations are represented by each expression.
Note, I will express the set of free indices by $\mathcal{F}$ and the set of dummy indices as $\mathcal{D}$, and I will use the original index names.

(a) $A^\alpha B_\beta = 5 \implies A^\beta B_\alpha = 5$ (16 equations, $\mathcal{F} = \{\alpha, \beta\}$, $\mathcal{D} = \emptyset$)

(b) $A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_\nu A^\nu \implies A^{\bar{\nu}} = \Lambda^{\bar{\nu}}{}_\mu A^\mu$ (4 equations, $\mathcal{F} = \{\bar{\mu}\}$, $\mathcal{D} = \{\nu\}$).

(c) $T^{\alpha\mu\lambda} A_\mu C_\lambda{}^\gamma = D^{\gamma\alpha} \implies T^{\eta\phi\theta} A_\phi C_\theta{}^\zeta = D^{\zeta\eta}$ (16 equations, $\mathcal{F} = \{\alpha, \gamma\}$, $\mathcal{D} = \{\mu, \lambda\}$)

(d) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = G_{\mu\nu} \implies R_{\chi\epsilon} - \frac{1}{2} g_{\chi\epsilon} = G_{\chi\epsilon}$ (16 equations, $\mathcal{F} = \{\mu, \nu\}$, $\mathcal{D} = \emptyset$)

4 Given vectors $\vec{A} \to_{\mathcal{O}} (5, -1, 0, 1)$ and $\vec{B} \to_{\mathcal{O}} (-2, 1, 1, -6)$, find the components in $\mathcal{O}$ of

(a) $-6\vec{A} \to_{\mathcal{O}} (-30, 6, 0, -6)$

(b) $3\vec{A} + \vec{B} \to_{\mathcal{O}} (13, -2, 1, -3)$

(c) $-6\vec{A} + 3\vec{B} \to_{\mathcal{O}} (-36, 9, 3, -24)$

6 Draw a spacetime diagram from $\mathcal{O}$'s reference frame. There are two other frames, $\bar{\mathcal{O}}$ and $\bar{\bar{\mathcal{O}}}$, which are each moving with velocity 0.6 in the $+x$ direction from each respective frame. Plot each frame's basis vectors, as observed by $\mathcal{O}$.

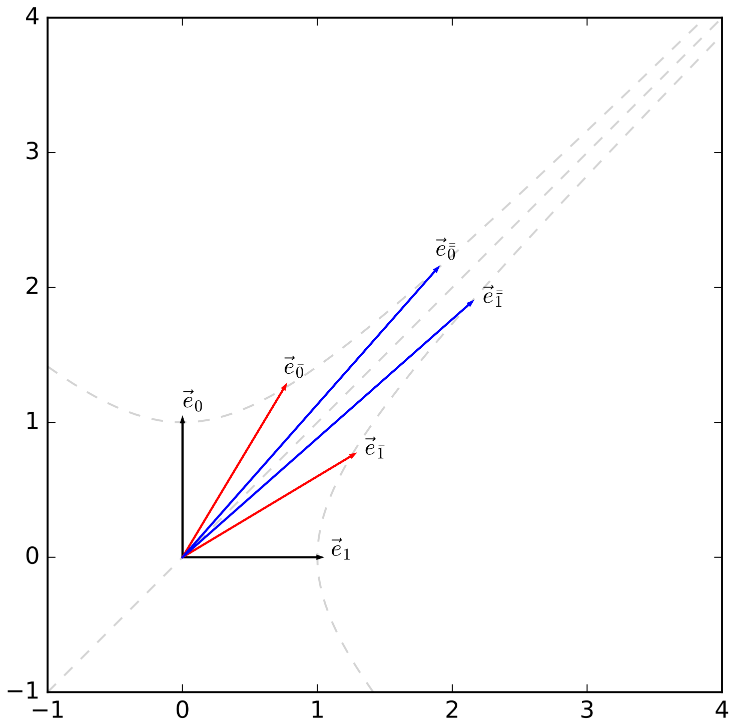


Figure 2.1: Exercise 6

See Figure 2.1.

9 Prove, by writing out all the terms that

$$\sum_{\bar{\alpha}=0}^{3}\left(\sum_{\beta=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)=\sum_{\beta=0}^{3}\left(\sum_{\bar{\alpha}=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right)$$

$$\begin{aligned}
\sum_{\bar{\alpha}=0}^{3}\left(\sum_{\beta=0}^{3}\Lambda^{\bar{\alpha}}{}_{\beta}A^{\beta}\vec{e}_{\bar{\alpha}}\right) &= \sum_{\bar{\alpha}=0}^{3}\left(\Lambda^{\bar{\alpha}}{}_{0}A^{0}\vec{e}_{\bar{\alpha}}+\Lambda^{\bar{\alpha}}{}_{1}A^{1}\vec{e}_{\bar{\alpha}}+\Lambda^{\bar{\alpha}}{}_{2}A^{2}\vec{e}_{\bar{\alpha}}+\Lambda^{\bar{\alpha}}{}_{3}A^{3}\vec{e}_{\bar{\alpha}}\right)\\
&= \Lambda^{\bar{0}}{}_{0}A^{0}\vec{e}_{\bar{0}}+\Lambda^{\bar{0}}{}_{1}A^{1}\vec{e}_{\bar{0}}+\Lambda^{\bar{0}}{}_{2}A^{2}\vec{e}_{\bar{0}}+\Lambda^{\bar{0}}{}_{3}A^{3}\vec{e}_{\bar{0}}\\
&+ \Lambda^{\bar{1}}{}_{0}A^{0}\vec{e}_{\bar{1}}+\Lambda^{\bar{1}}{}_{1}A^{1}\vec{e}_{\bar{1}}+\Lambda^{\bar{1}}{}_{2}A^{2}\vec{e}_{\bar{1}}+\Lambda^{\bar{1}}{}_{3}A^{3}\vec{e}_{\bar{1}}\\
&+ \Lambda^{\bar{2}}{}_{0}A^{0}\vec{e}_{\bar{2}}+\Lambda^{\bar{2}}{}_{1}A^{1}\vec{e}_{\bar{2}}+\Lambda^{\bar{2}}{}_{2}A^{2}\vec{e}_{\bar{2}}+\Lambda^{\bar{2}}{}_{3}A^{3}\vec{e}_{\bar{2}}\\
&+ \Lambda^{\bar{3}}{}_{0}A^{0}\vec{e}_{\bar{3}}+\Lambda^{\bar{3}}{}_{1}A^{1}\vec{e}_{\bar{3}}+\Lambda^{\bar{3}}{}_{2}A^{2}\vec{e}_{\bar{3}}+\Lambda^{\bar{3}}{}_{3}A^{3}\vec{e}_{\bar{3}}\\
&= \Lambda^{\bar{0}}{}_{0}A^{0}\vec{e}_{\bar{0}}+\Lambda^{\bar{1}}{}_{0}A^{0}\vec{e}_{\bar{1}}+\Lambda^{\bar{2}}{}_{0}A^{0}\vec{e}_{\bar{2}}+\Lambda^{\bar{3}}{}_{0}A^{0}\vec{e}_{\bar{3}}\\
&+ \Lambda^{\bar{0}}{}_{1}A^{1}\vec{e}_{\bar{0}}+\Lambda^{\bar{1}}{}_{1}A^{1}\vec{e}_{\bar{1}}+\Lambda^{\bar{2}}{}_{1}A^{1}\vec{e}_{\bar{2}}+\Lambda^{\bar{3}}{}_{1}A^{1}\vec{e}_{\bar{3}}\\
&+ \Lambda^{\bar{0}}{}_{2}A^{2}\vec{e}_{\bar{0}}+\Lambda^{\bar{1}}{}_{2}A^{2}\vec{e}_{\bar{1}}+\Lambda^{\bar{2}}{}_{2}A^{2}\vec{e}_{\bar{2}}+\Lambda^{\bar{3}}{}_{2}A^{2}\vec{e}_{\bar{3}}
\end{aligned}$$

$$\begin{aligned}
&+ \Lambda^{\bar{0}}{}_3 A^3 \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_3 A^3 \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_3 A^3 \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_3 A^3 \vec{e}_{\bar{3}} \\
&= \sum_{\beta=0}^{3} \left(\Lambda^{\bar{0}}{}_\beta A^\beta \vec{e}_{\bar{0}} + \Lambda^{\bar{1}}{}_\beta A^\beta \vec{e}_{\bar{1}} + \Lambda^{\bar{2}}{}_\beta A^\beta \vec{e}_{\bar{2}} + \Lambda^{\bar{3}}{}_\beta A^\beta \vec{e}_{\bar{3}} \right) \\
&= \sum_{\beta=0}^{3} \left(\sum_{\bar{\alpha}=0}^{3} \Lambda^{\bar{\alpha}}{}_\beta A^\beta \vec{e}_{\bar{\alpha}} \right)
\end{aligned}$$

11 Let $\Lambda^{\bar{\alpha}}{}_\beta$ be the matrix of the Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$, given in Equation 1.12. Let $\vec{A}$ be an arbitrary vector with components (A^0, A^1, A^2, A^3) in frame $\mathcal{O}$.

(a) Write down the matrix of $\Lambda^\nu{}_{\bar{\mu}}(-v)$.

Intuitively, it should appear the same as $\Lambda^{\bar{\alpha}}{}_\beta$, but with the negative signs removed. More rigorously, it is given by the matrix inverse of $\Lambda^{\bar{\alpha}}{}_\beta$, as their product should be the identity matrix. I have used a computer algebra system (Wolfram Alpha) to take the inverse of this matrix symbolically, confirming my suspicion:

$$\Lambda^\nu{}_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Find $A^{\bar{\alpha}}$ for all $\bar{\alpha}$.

$$\begin{aligned}
A^{\bar{\alpha}} &= \Lambda^{\bar{\alpha}}{}_\beta A^\beta \\
A^{\bar{0}} &= \gamma(A^0 - vA^1) \\
A^{\bar{1}} &= \gamma(A^1 - vA^0) \\
A^{\bar{2}} &= A^2 \\
A^{\bar{3}} &= A^3
\end{aligned}$$

(c) Verify Equation 2.18 by performing the sum for all values of ν and α.

To simplify things, I do this via matrix multiplication

$$\Lambda^{\bar{\alpha}}{}_\beta(v)\Lambda^\nu{}_{\bar{\mu}}(-v) = \begin{pmatrix} \gamma^2 - v^2\gamma^2 & v\gamma^2 - v\gamma^2 & 0 & 0 \\ v\gamma^2 - v\gamma^2 & \gamma^2 - v^2\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2(1-v^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-v^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\nu}{}_{\alpha}$$

(d) Write down the Lorentz transformation matrix from $\bar{\mathcal{O}}$ to $\mathcal{O}$, justifying each term.

It should just be $\Lambda^{\nu}{}_{\bar{\mu}}(-v)$. I'm not sure what else to say at this point.

(e) Using the result from part (d), find A^{β} from $A^{\bar{\alpha}}$. How does this relate to Equation 2.18?

$$\Lambda^{\beta}{}_{\bar{\alpha}}A^{\bar{\alpha}} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma(A^0 - vA^1) \\ \gamma(A^1 - vA^0) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma^2(A^0 - vA^1) + v\gamma^2(A^1 - vA^0) + 0 + 0 \\ v\gamma^2(A^0 - vA^1) + \gamma^2(A^1 - vA^0) + 0 + 0 \\ A^2 \\ A^3 \end{pmatrix}$$

$$= \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) + A^1(v\gamma^2 - v\gamma^2) \\ A^0(v\gamma^2 - v^2\gamma^2) + A^1(\gamma^2 - v\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0(\gamma^2 - v^2\gamma^2) \\ A^1(\gamma^2 - v^2\gamma^2) \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = A^{\beta}$$

Since $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta}(v)$, this goes to show that $\Lambda^{\nu}{}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}{}_{\alpha}(-v)A^{\lambda} = A^{\lambda} \implies \Lambda^{\nu}{}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}{}_{\alpha}(-v) = \delta^{\nu}{}_{\alpha}$.

(f) Verify in the same manner as (c) that

$$\Lambda^{\nu}{}_{\bar{\beta}}(v)\Lambda^{\bar{\alpha}}{}_{\nu}(-v) = \delta^{\bar{\alpha}}{}_{\bar{\beta}}$$

My matrix multiplication approach will just give me the same result as before. Perhaps another approach was intended?

(g) Establish that

$$\vec{e}_{\alpha} = \Lambda^{\bar{\beta}}{}_{\alpha}\vec{e}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}\Lambda^{\nu}{}_{\bar{\beta}}\vec{e}_{\nu} = \delta^{\nu}{}_{\alpha}\vec{e}_{\nu}$$

$$A^{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}A^{\alpha} = \Lambda^{\bar{\beta}}{}_{\alpha}\Lambda^{\alpha}{}_{\bar{\mu}}A^{\bar{\mu}} = \delta^{\bar{\beta}}{}_{\bar{\mu}}A^{\bar{\mu}}$$

14 The following matrix gives a Lorentz transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$:

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a) What is the velocity of $\bar{\mathcal{O}}$ relative to $\mathcal{O}$?

This would correspond to a Lorentz boost along the z-axis, meaning

$$\Lambda^{\bar{\alpha}}{}_{\beta}(v) = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix},$$

and thus we have $\gamma = 1.25$ and $-v\gamma = 0.75$. Solving for v, we get

$$-v\gamma = \frac{3}{4} \implies v = -\frac{3}{4\gamma} = -\frac{3 \cdot 4}{4 \cdot 5} = -\frac{3}{5}.$$

So $\bar{\mathcal{O}}$ is moving with speed 0.6 relative to the $-z$-axis of $\mathcal{O}$.

(b) What is the inverse matrix to the given one?

Numerically, it comes out to be

$$\begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix},$$

which makes sense, when you consider that the inverse matrix should be a Lorentz transformation with the velocity negated.

(c) Find the components in $\mathcal{O}$ of $\vec{A} \to_{\bar{\mathcal{O}}} (1, 2, 0, 0)$.

$$\vec{A} \underset{\mathcal{O}}{\rightarrow} \begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 2 \\ 0 \\ -0.75 \end{pmatrix}$$

15

(a) Compute the four-velocity components in $\mathcal{O}$ of a particle whose speed is v in the $+x$-direction relative to $\mathcal{O}$, using the Lorentz transformation.

$$\begin{aligned}
\vec{U} &= \vec{e}_{\bar{0}} \\
U^\alpha &= \Lambda^\alpha{}_{\bar{\beta}}(\vec{e}_{\bar{0}})^{\bar{\beta}} = \Lambda^\alpha_{\bar{0}}, \\
U^0 &= \gamma \\
U^1 &= v\gamma \\
U^2 &= U^3 = 0
\end{aligned}$$

(b) Generalize to arbitrary velocities $\boldsymbol{v}$, where $|v| < 1$.

$$\Lambda^\alpha{}_{\bar{\beta}}(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

$$U^0 = \gamma \quad U^1 = \gamma v_x \quad U^2 = \gamma v_y \quad U^3 = \gamma v_z$$

(c) Use this result to express $\boldsymbol{v}$ as a function of the components $\{U^\alpha\}$.

$$\begin{aligned}
\boldsymbol{v} &= v_x \vec{e}_1 + v_y \vec{e}_2 + v_z \vec{e}_3 \\
v_i &= \frac{U^i}{\gamma} \\
\boldsymbol{v} &= \frac{1}{\gamma} U^i \vec{e}_i
\end{aligned}$$

(d) Find the three-velocity $\boldsymbol{v}$ of a particle with four-velocity components $(2, 1, 1, 1)$.

$U^0 = \gamma = 2$, and $U^i = 1$, so

$$\boldsymbol{v} = \frac{1}{2}\vec{e}_i$$

17

Not sure how to approach this problem.

(a) Prove that any timelike vector $\vec{U}$ for which $U^0 > 0$ and $\vec{U} \cdot \vec{U} = -1$ is the four-velocity of *some* world line.

(b) Use this to prove that for any timelike vector $\vec{V}$ there is a Lorentz frame in which the $\vec{V}$ has zero spatial components.

19 A body is uniformly accelerated if the four-vector $\vec{a}$ has constant spatial direction and magnitude, $\vec{a} \cdot \vec{a} = \alpha^2 \geqslant 0$.

(a) Show that this implies the components of $\vec{a}$ in the body's MCRF are all constant, and that these are equivalent to the Galilean "acceleration".

We normalize the vector $\vec{a}$ by dividing each of its terms by the magnitude of the vector, so

$$\frac{a^\lambda}{\alpha}.$$

Since α is constant, and also the *direction* is constant, this means that the above expression is *also* constant, as the normalized components tell you about the direction. If we multiply a constant by a constant, we should still get a constant, so we multiply the above expression by α, getting a^λ to be constant.

In the MCRF of an object, $\mathrm{d}\tau = \mathrm{d}t$, and so we can write

$$\vec{a} = \frac{\mathrm{d}\vec{U}}{\mathrm{d}t} = \left(0, \frac{\mathrm{d}U^1}{\mathrm{d}t}, \frac{\mathrm{d}U^2}{\mathrm{d}t}, \frac{\mathrm{d}U^3}{\mathrm{d}t}\right),$$

which is analogous to the Galilean acceleration.

(b) A body is uniformly accelerated with $\alpha = 10\,\mathrm{m/s^2}$. It starts from rest, and falls for a time t. Find its speed as a function of t, and find the time to reach $v = 0.999$.

$$\begin{aligned}
\vec{U} &\underset{\text{MCRF}}{\to} (1, 0, 0, 0) \\
&\underset{\mathcal{O}}{\to} (\gamma, \gamma v, 0, 0) \\
\frac{\mathrm{d}\vec{U}}{\mathrm{d}\tau} &\underset{\text{MCRF}}{\to} (0, \alpha, 0, 0) \\
&\underset{\mathcal{O}}{\to} (\gamma, \gamma\alpha, 0, 0) \\
U^x &= \int_0^t \frac{\mathrm{d}U^x}{\mathrm{d}\tau}\mathrm{d}\tau = \int_0^t \gamma\alpha\frac{\mathrm{d}t}{\gamma} = \int_0^t \alpha\,\mathrm{d}t = \alpha t \\
&= \gamma v = \frac{v}{\sqrt{1-v^2}} \\
v^2 &= (\alpha t)^2(1-v^2) = (\alpha t)^2 - (\alpha t v)^2 \\
v^2(1+(\alpha t)^2) &= (\alpha t)^2 \\
v^2 &= \frac{(\alpha t)^2}{1+(\alpha t)^2} \implies v = \sqrt{\frac{(\alpha t)^2}{1+(\alpha t)^2}}
\end{aligned}$$

To find the time to reach $v = 0.999$, we go back to the expression $\gamma v = \alpha t$, solve for t, and substitute for v and α. Note that in natural units, $\alpha = 10\,\mathrm{m/s^2}c^{-2} \approx 1.11 \times 10^{-16}\,\mathrm{m}^{-1}$

$$t = \frac{v}{\alpha\sqrt{1-v^2}} = \frac{0.999}{1.11 \times 10^{-16}\,\mathrm{m}^{-1}\sqrt{1-0.999^2}} \approx 2.01 \times 10^{17}\,\mathrm{m}.$$

24 Show that a positron and electron cannot annihilate to form a single photon, but they can annihilate to form two photons.

We consider the center of momentum frame, where $\sum \vec{p}_{(i)} \to_{\text{CM}} (E_{\text{total}}, 0, 0, 0)$. Without loss of generality, we assume that the velocities of the two particles are equal and opposite, such that

$$\vec{p}_{e^+} \to_{\text{CM}} m_e(\gamma, \gamma v, 0, 0), \qquad \vec{p}_{e^-} \to_{\text{CM}} m_e(\gamma, -\gamma v, 0, 0).$$

The photon they create will have to have a momentum of $\vec{p}_{\gamma,\text{single}} \to_{\text{CM}} (h\nu, h\nu, 0, 0)$. By conservation of four-momentum, we have

$$\begin{aligned}
\vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma,\text{single}} \\
(\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= \vec{p}_{\gamma,\text{single}} \cdot \vec{p}_{\gamma,\text{single}} \\
(\vec{p}_{e^+} \cdot \vec{p}_{e^+}) + (\vec{p}_{e^-} \cdot \vec{p}_{e^-}) + (\vec{p}_{e^+} \cdot \vec{p}_{e^-}) &= 0 \\
-m_e^2 - m_e^2 - m_e^2 &= 0 \implies m_e = 0!
\end{aligned}$$

Since we know that m_e is in fact non-zero, this cannot possibly happen.

Now consider the scenario wherein two photons are created, moving in opposite directions. Then they would have momenta: $\vec{p}_{\gamma,1} \to_{\text{CM}} (h\nu, h\nu, 0, 0)$ and $\vec{p}_{\gamma,2} \to_{\text{CM}} (h\nu, -h\nu, 0, 0)$. Invoking conservation of four-momentum as before, we get

$$\begin{aligned}
\vec{p}_{e^+} + \vec{p}_{e^-} &= \vec{p}_{\gamma,1} + \vec{p}_{\gamma,2} \\
(\vec{p}_{e^+} + \vec{p}_{e^-}) \cdot (\vec{p}_{e^+} + \vec{p}_{e^-}) &= (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \cdot (\vec{p}_{\gamma,1} + \vec{p}_{\gamma,2}) \\
-3m_e^2 &= (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,1}) + (\vec{p}_{\gamma,1} \cdot \vec{p}_{\gamma,2}) + (\vec{p}_{\gamma,2} \cdot \vec{p}_{\gamma,2}) \\
&= 0 + (-h^2\nu^2 - h^2\nu^2) + 0 = -2h^2\nu^2,
\end{aligned}$$

so we end up with $3m_e^2 = 2h^2\nu^2$, meaning two photons are produced with $E^2 = \frac{3}{2}m_e^2$, which is entirely reasonable.

25

(a) Consider a frame $\bar{\mathcal{O}}$ moving with a speed v along the x-axis of $\mathcal{O}$. Now consider a photon moving at an angle θ from $\mathcal{O}$'s x-axis. Find the ratio of its frequency in $\bar{\mathcal{O}}$ and in $\mathcal{O}$.

We must first construct the particle's four-momentum. In the case where the photon was moving along the x-axis (see Section 2.7), it had been found that the four-momentum was

$$\vec{p} \underset{\mathcal{O}}{\to} (E, E, 0, 0),$$

as this satisfied

$$\vec{p} \cdot \vec{p} = -E^2 + E^2 = 0. \qquad \text{(Schutz 2.37)}$$

Now that the photon is moving at an angle θ from the x-axis, we need to redistribute the 3-momentum accordingly. No specification was given as photon's angle in the y- or z-axis, so without loss of generality, I assume it is constrained to the x-y plane. This means we can write the four-momentum as

$$\vec{p} \underset{\mathcal{O}}{\to} (E, E\cos\theta, E\sin\theta, 0),$$

which you can easily confirm satisfies $\vec{p} \cdot \vec{p} = 0$.

Now we may apply the Lorentz transformation $\Lambda^{\bar{0}}{}_{\alpha}(v)$ to find the photon's energy as observed by $\bar{\mathcal{O}}$, and from that the frequency.

$$p^{\bar{0}} = \bar{E} = \Lambda^{\bar{0}}{}_{\alpha} p^{\alpha} = \gamma p^0 - v\gamma p^1 + 0 + 0 = \gamma E - v\gamma E \cos\theta$$
$$\implies h\bar{\nu} = \gamma h\nu - v\gamma h\nu \cos\theta$$
$$\implies \frac{\bar{\nu}}{\nu} = \gamma - v\gamma\cos\theta = \frac{1 - v\cos\theta}{\sqrt{1-v^2}}$$

(b) Even when the photon moves perpendicular to the x-axis ($\theta = \pi/2$) there is a frequency shift. This is the *transverse Doppler shift*, which is a result of time dilation. At which angle θ must the photon move such that there is no Doppler shift between $\mathcal{O}$ and $\bar{\mathcal{O}}$?

To do this, we simply set $\bar{\nu}/\nu = 1$, and solve for θ.

$$1 = \frac{1 - v\cos\theta}{\sqrt{1-v^2}} \implies \cos\theta = 1 - \sqrt{1-v^2}$$
$$\implies \theta = \pm\arccos\left(1 - \sqrt{1-v^2}\right)$$

(c) Now use Equations 2.35 and 2.38 to find $\bar{\nu}/\nu$.

Recall that $\vec{U} \rightarrow_{\mathcal{O}} (\gamma, v\gamma, 0, 0)$. Using Equation 2.35 we have

$$\bar{E} = h\bar{\nu} = -(E, E\cos\theta, E\sin\theta, 0) \cdot (\gamma, v\gamma, 0, 0)$$
$$= -(-(E\gamma) + E\gamma v\cos\theta) = E\gamma(1 - v\cos\theta) = h\nu\gamma(1 - v\cos\theta)$$
$$\frac{\bar{\nu}}{\nu} = \frac{1 - v\cos\theta}{\sqrt{1-v^2}}$$

26 Calculate the energy required to accelerate a particle of rest mass $m > 0$ from speed v to speed $v + \delta v$ ($\delta v \ll v$), to first order in δv. Show that it would take infinite energy to accelerate to c.

From the four-momentum we have $E_v = m\gamma$, and from that

$$E_{v+\delta v} = \frac{m}{\sqrt{1-(v+\delta v)^2}}.$$

If we do a Taylor expansion on $(1-(v+\delta v)^2)^{-1/2}$ we get

$$\frac{1}{\sqrt{1-v^2}} + \frac{v\,\delta v}{(1-v^2)^{3/2}} + \mathcal{O}\left(v^2\right),$$

so

$$E_{v+\delta v} \approx \frac{m}{\sqrt{1-v^2}} + \frac{mv\,\delta v}{(1-v^2)^{3/2}}$$
$$\Delta E = E_{v+\delta v} - E_v \approx \frac{mv\,\delta v}{(1-v^2)^{3/2}} = m\gamma^3 v\,\delta v\,.$$

As $v \to c$, $\gamma \to \infty$ and therefore $\Delta E \to \infty$.

30 A rocket ship has four-velocity $\vec{U} \rightarrow_{\mathcal{O}} (2, 1, 1, 1)$, and it passes a cosmic ray with four-momentum $\vec{p} \rightarrow \mathcal{O}(300, 299, 0, 0) \times 10^{-27}$kg. Compute the energy of the ray as measured by the rocket, using two different methods.

(a) Find the Lorentz transformation from $\mathcal{O}$ to the rocket's MCRF, and from that find the components $p^{\bar{\alpha}}$. The Lorentz transformation for a boost in the x, y, and z directions is given by

$$\Lambda^{\bar{\beta}}{}_{\alpha} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix}.$$

If we write out the terms of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v_x & \gamma v_y & \gamma v_z \\ \gamma v_x & \gamma & 0 & 0 \\ \gamma v_y & 0 & \gamma & 0 \\ \gamma v_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

then we are left with a system of equations

$$\begin{aligned} 1 &= \gamma(2 + v_x + v_y + v_z), \\ 0 &= \gamma(2v_x + 1), \\ 0 &= \gamma(2v_y + 1), \\ 0 &= \gamma(2v_z + 1). \end{aligned}$$

Since γ may never be zero, we divide the last 3 terms by γ to obtain

$$2v_i + 1 = 0 \implies v_i = -\frac{1}{2},$$

and plugging into the first equation gives $\gamma = 2$. From this we see that our Lorentz transformation matrix is

$$\Lambda^{\bar{\beta}}{}_{\alpha} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

Now to find the energy as observed by the rocket, we need to find $\bar{E} = p^{\bar{0}}$

$$\begin{aligned} p^{\bar{0}} &= \Lambda^{\bar{0}}{}_{\alpha} p^{\alpha} = 2p^0 - p^1 - p^2 - p^3 \\ &= (2 \cdot 300 - 1 \cdot 299 - 1 \cdot 0 - 1 \cdot 0) \times 10^{-27}\text{kg} = 3.01 \times 10^{-25}\,\text{kg} = \bar{E} \end{aligned}$$

(b) Use Schutz's Equation 2.35.

$$
\begin{aligned}
\bar{E} &= -\vec{p} \cdot \vec{U}_{\text{obs}} = -(-(300 \cdot 2) + (299 \cdot 1) + (0 \cdot 1) + (0 \cdot 1)) \times 10^{-27}\text{kg} \\
&= 3.01 \times 10^{-25}\,\text{kg}
\end{aligned}
$$

(c) Which is quicker? Why?

Using Equation 2.35 was *much* quicker, as it was derived to handle this special case.

32 Consider a particle with charge e and mass m, which begins at rest, but scatters a photon with frequency ν_i (Compton scattering). The photon comes off at an angle θ from the direction of the initial photon's path. Use conservation of four-momentum to find the scattered photon's frequency, ν_f.

We will invoke: conservation of four-momentum and $\vec{p} \cdot \vec{p} = -m^2$. $\vec{p}_i$ and $\vec{p}_f$ denote the initial and final photon, and $\vec{p}_e$ and $\vec{p}_{e'}$ denote the electron before and after collision.

$$
\begin{aligned}
\vec{p}_i &\underset{\mathcal{O}}{\rightarrow} (E_i, E_i, 0, 0) \\
\vec{p}_e &\underset{\mathcal{O}}{\rightarrow} (m, 0, 0, 0) \\
\vec{p}_f &\underset{\mathcal{O}}{\rightarrow} (E_f, E_f \cos\theta, E_f \sin\theta, 0) \\
\vec{p}_i + \vec{p}_e &= \vec{p}_f + \vec{p}_{e'} \\
\vec{p}_{e'} &= \vec{p}_i + \vec{p}_e - \vec{p}_f \\
\vec{p}_{e'} \cdot \vec{p}_{e'} &= (\vec{p}_i + \vec{p}_e - \vec{p}_f) \cdot (\vec{p}_i + \vec{p}_e - \vec{p}_f) \\
-m^2 &= \vec{p}_i \cdot \vec{p}_i + \vec{p}_e \cdot \vec{p}_e + \vec{p}_f \cdot \vec{p}_f + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\
&= 0 - m^2 + 0 + 2(\vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f) \\
0 &= \vec{p}_i \cdot \vec{p}_i - \vec{p}_i \cdot \vec{p}_f - \vec{p}_e \cdot \vec{p}_f \\
&= -E_i m - (-E_i E_f + E_i E_f \cos\theta) + E_f m \\
&= m(E_f - E_i) + E_i E_f (1 - \cos\theta) \\
m(E_i - E_f) &= E_i E_f (1 - \cos\theta) \\
mh(\nu_i - \nu_f) &= h^2 \nu_i \nu_f (1 - \cos\theta) \\
\frac{\nu_i - \nu_f}{\nu_i \nu_f} &= h \frac{1 - \cos\theta}{m} \\
\frac{1}{\nu_f} - \frac{1}{\nu_i} &= h \frac{1 - \cos\theta}{m} \\
\frac{1}{\nu_f} &= \frac{1}{\nu_i} + h \frac{1 - \cos\theta}{m}
\end{aligned}
$$

Chapter 3

Tensor analysis in special relativity

3.3 The $\binom{0}{1}$ tensors: one-forms

The symbol $\tilde{}$ is used to denote a one-form, as $\vec{}$ is used to denote a vector. So $\tilde{p}$ is a one-form, or a type $\binom{0}{1}$ tensor.

Normal one-forms

Let $\mathcal{S}$ be some surface.
$\forall \vec{V}$ tangent to $\mathcal{S}$, $\tilde{p}(\vec{V}) = 0 \implies \tilde{p}$ is normal to $\mathcal{S}$.
Furthermore, if $\mathcal{S}$ is a *closed* surface & $\tilde{p}$ is normal to $\mathcal{S}$ & $\forall \vec{U}$ pointing outwards from $\mathcal{S}$, $\tilde{p}(\vec{U}) > 0 \implies \tilde{p}$ is an outward normal one-form.

3.5 Metric as a mapping of vectors into one-forms

Normal vectors and unit normal one-forms

$\vec{V}$ is normal to a surface if $\tilde{V}$ is normal to the surface. They are said to be *unit normal* if their magnitude is ± 1, so $\vec{V}^2 = \tilde{V}^2 = \pm 1$.

- A time-like unit normal has magnitude -1
- A space-like unit normal has magnitude $+1$
- A null normal cannot be a unit normal, because $\vec{V}^2 = \tilde{V}^2 = 0$

3.10 Exercises

3

(a)

$$\begin{aligned}\tilde{p}(A^\alpha \vec{e}_\alpha) &= A^\alpha \tilde{p}(\vec{e}_\alpha) = \tilde{p}(A^0\vec{e}_0 + A^1\vec{e}_1 + A^2\vec{e}_2 + A^3\vec{e}_3) \\ &= A^0\tilde{p}(\vec{e}_0) + A^1\tilde{p}(\vec{e}_1 + A^2\tilde{p}(\vec{e}_2) + A^3\tilde{p}(\vec{e}_3 = A^\alpha \tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha \in \mathbb{R}\end{aligned}$$

(b)

$$\begin{aligned}\tilde{p} &\underset{\mathcal{O}}{\rightarrow} (-1, 1, 2, 0) \\ \vec{A} &\underset{\mathcal{O}}{\rightarrow} (2, 1, 0, -1) \\ \vec{B} &\underset{\mathcal{O}}{\rightarrow} (0, 2, 0, 0)\end{aligned}$$

$$\begin{aligned}\tilde{p}(\vec{A}) &= -2 + 1 + 0 + 0 = -1 \\ \tilde{p}(\vec{B}) &= 0 + 2 + 0 + 0 = 2 \\ \tilde{p}(\vec{A} - 3\vec{B}) &= \tilde{p}(\vec{A}) - 3\tilde{p}(\vec{B}) = -1 - 3 \cdot 2 = -7\end{aligned}$$

4 Given the following vectors

$$\vec{A} \underset{\mathcal{O}}{\rightarrow} (2, 1, 1, 0) \qquad \vec{B} \underset{\mathcal{O}}{\rightarrow} (1, 2, 0, 0)$$
$$\vec{C} \underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1) \qquad \vec{D} \underset{\mathcal{O}}{\rightarrow} (-3, 2, 0, 0)$$

(Note that all parts were done with the assistance of `numpy`.)

(a) Show that they are linearly independent.

We do this by constructing a matrix, $\mathbf{X}$, whose columns correspond to the four vectors. If the determinant of $\mathbf{X}$ is non-zero, then that means the vectors are linearly independent.

$$\det(\mathbf{X}) = \det \begin{pmatrix} 2 & 1 & 0 & -3 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -8$$

(b) Find the components of $\tilde{p}$ if

$$\tilde{p}(\vec{A}) = 1, \quad \tilde{p}(\vec{B}) = -1, \quad \tilde{p}(\vec{C}) = -1, \quad \tilde{p}(\vec{D}) = 0$$

We do this by observing that $\tilde{p} = A^\alpha p_\alpha$, and so we have a system of four equations, which we can write in matrix form as

$$\begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\\ \vec{D}\end{pmatrix}\tilde{p} = \begin{pmatrix}1\\ -1\\ -1\\ 0\end{pmatrix}$$

$$\implies \tilde{p} = \begin{pmatrix}\vec{A}\\ \vec{B}\\ \vec{C}\\ \vec{D}\end{pmatrix}^{-1} \begin{pmatrix}1\\ -1\\ -1\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{4}\\ -\frac{3}{8}\\ +\frac{15}{8}\\ -\frac{23}{8}\end{pmatrix}.$$

(c) Find $\tilde{p}(\vec{E})$, where $\vec{E} \to_{\mathcal{O}} (1, 1, 0, 0)$.

$$\tilde{p}(\vec{E}) = p_\alpha E^\alpha = -\frac{5}{8}$$

(d) Determine whether $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$ are linearly independent.

We do this by first setting up a system of equations for each of $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$, as was done for $\tilde{p}$, and solving. I will refer to the matrix whose rows were $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ as $\mathbf{X}$.

$$\mathbf{X}\tilde{q} = \begin{pmatrix}+0\\ +0\\ +1\\ -1\end{pmatrix} \qquad \mathbf{X}\tilde{r} = \begin{pmatrix}+2\\ +0\\ +0\\ +0\end{pmatrix} \qquad \mathbf{X}\tilde{s} = \begin{pmatrix}-1\\ -1\\ +0\\ +0\end{pmatrix}$$

$$\tilde{q} = \begin{pmatrix}+\frac{1}{4}\\ -\frac{1}{8}\\ -\frac{3}{8}\\ +\frac{11}{8}\end{pmatrix} \qquad \tilde{r} = \begin{pmatrix}+0\\ +0\\ +2\\ +2\end{pmatrix} \qquad \tilde{s} = \begin{pmatrix}-\frac{1}{4}\\ -\frac{3}{8}\\ -\frac{1}{8}\\ +\frac{1}{8}\end{pmatrix}$$

Now if the matrix whose columns are comprised of $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, and $\tilde{s}$ has a non-zero determinant, then the four covectors must be linearly independent.

$$\det\begin{pmatrix}\tilde{p} & \tilde{q} & \tilde{r} & \tilde{s}\end{pmatrix} = \frac{1}{4},$$

and so they are indeed linearly independent.

6

(a) Show that $\tilde{p} \neq \tilde{p}(\vec{e}_\alpha)\tilde{\lambda}^\alpha$ for arbitrary $\tilde{p}$.

Let us choose $\tilde{p} \to_{\mathcal{O}} (0, 1, e, \pi)$, as a counter-example.

$$\begin{aligned} p_\alpha \tilde{\lambda}^\alpha &\underset{\mathcal{O}}{\to} 0 \cdot (1, 1, 0, 0) + 1 \cdot (1, -1, 0, 0) + e \cdot (0, 0, 1, -1) + \pi \cdot (0, 0, 1, 1) \\ &\underset{\mathcal{O}}{\to} (1, -1, e + \pi, 0) \underset{\mathcal{O}}{\not\to} \tilde{p} \end{aligned}$$

(b) $\tilde{p} \to_{\mathcal{O}} (1, 1, 1, 1)$. Find l_α such that

$$\tilde{p} = l_\alpha \tilde{\lambda}^\alpha$$

We may do this with a simple matrix inversion. We define $\mathbf{\Lambda}$ to be the matrix whose rows are formed by $\tilde{\lambda}^\alpha$.

$$\mathbf{\Lambda} l = p \implies l = \mathbf{\Lambda}^{-1} p = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

8 Draw the basis one-forms $\tilde{\mathrm{d}}t$ and $\tilde{\mathrm{d}}x$ of frame $\mathcal{O}$.

They are

$$\begin{aligned} \tilde{\mathrm{d}}t &\underset{\mathcal{O}}{\to} (1, 0, 0, 0), \\ \tilde{\mathrm{d}}x &\underset{\mathcal{O}}{\to} (0, 1, 0, 0), \end{aligned}$$

and they are shown in Figure 3.1.

9 At the points $\mathcal{P}$ and $\mathcal{Q}$, estimate the components of the gradient $\tilde{\mathrm{d}}T$.

Recall that $\tilde{\mathrm{d}}T \to_{\mathcal{O}} \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)$, and so $\Delta T = \tilde{\mathrm{d}}T_\alpha x^\alpha = \tilde{\mathrm{d}}T_x \Delta x + \tilde{\mathrm{d}}T_y \Delta y$.

Now if we move only in the x direction from one of the points, we move some distance Δx, change our temperature by Δt, and $\Delta y = 0$. Likewise for a movement in the y direction. Thus we can say

$$\begin{aligned} \Delta T &= \tilde{\mathrm{d}}T_x \Delta x & \qquad \Delta T &= \tilde{\mathrm{d}}T_y \Delta y \\ \tilde{\mathrm{d}}T_x &= \frac{\Delta T}{\Delta x} & \qquad \tilde{\mathrm{d}}T_y &= \frac{\Delta T}{\Delta y} \end{aligned}$$

In Figure 3.2, from $\mathcal{P}$ I move a distance $\Delta x = 0.5$, which causes a temperature change of $\Delta T = -7$, giving $\tilde{\mathrm{d}}T_x = -14$. Then I move a distance $\Delta y = 0.5$ and get the same temperature change of $\Delta T = -7$, and so I conclude that at point $\mathcal{P}$, $\tilde{\mathrm{d}}T \to_{\mathcal{O}} (-14, -14)$.

At $\mathcal{Q}$, we are in a flat region where $T = 0$. If we move any non-zero distance Δx or Δy, so long as it does not cross the $T = 0$ isotherm, we have a $\Delta T = 0$, and thus $\tilde{\mathrm{d}}Tp \to_{\mathcal{O}} (0, 0)$.

13 Prove that $\tilde{\mathrm{d}}f$ is normal to surfaces of constant f.

If we move some small distance $\Delta x^\alpha = \epsilon$, then there will be no change in the value of f, and thus we can say $\partial f / \partial x^\alpha = 0$, so

$$\tilde{\mathrm{d}}f = \frac{\partial f}{\partial x^\alpha} \tilde{\mathrm{d}}x^\alpha = 0 \tilde{\mathrm{d}}x^\alpha = 0.$$

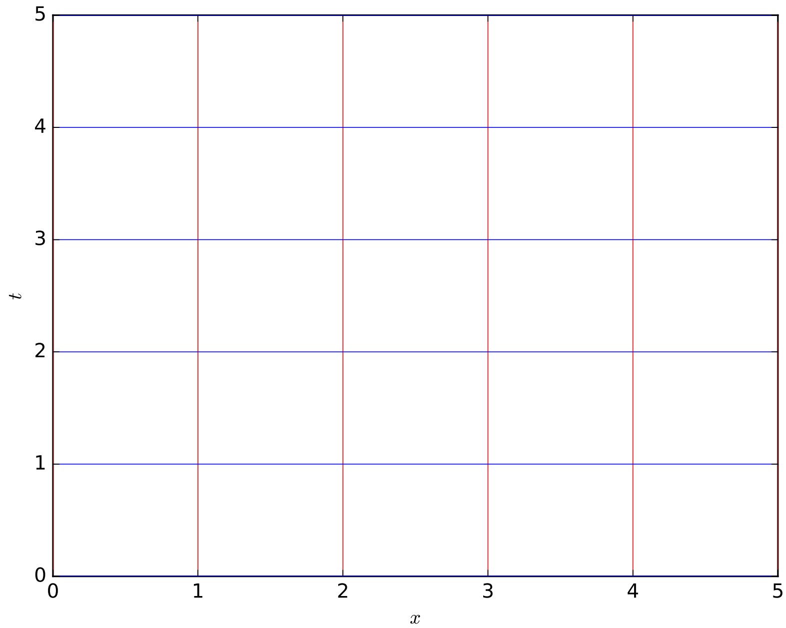


Figure 3.1: Problem 8: Basis one-forms of $\mathcal{O}$. $\tilde{\mathrm{d}}t$ is given in blue and $\tilde{\mathrm{d}}x$ in red.

Since $\tilde{\mathrm{d}}f$ is defined to be normal to a surface if it is zero on every tangent vector, we have shown that $\tilde{\mathrm{d}}f$ is normal to any surface of constant f.

14

$$\tilde{p} \underset{\mathcal{O}}{\rightarrow} (1, 1, 0, 0) \qquad \tilde{q} \underset{\mathcal{O}}{\rightarrow} (-1, 0, 1, 0)$$

Prove by giving two vectors $\vec{A}$ and $\vec{B}$ as arguments that $\tilde{p} \otimes \tilde{q} \neq \tilde{q} \otimes \tilde{p}$. Then find the components of $\tilde{p} \otimes \tilde{q}$.

$$\begin{aligned}(\tilde{p} \otimes \tilde{q})(\vec{A}, \vec{B}) &= \tilde{p}(\vec{A})\tilde{q}(\vec{B}) = A^\alpha p_\alpha B^\beta q_\beta = (A^0 + A^1)(-B^0 + B^2), \\ &= -A^0B^0 + A^0B^2 - A^1B^0 + A^1B^2 \\ (\tilde{q} \otimes \tilde{p})(\vec{A}, \vec{B}) &= \tilde{q}(\vec{A})\tilde{p}(\vec{B}) = A^\alpha q_\alpha B^\beta p_\beta = (-A^0 + A^2)(B^0 + B^1) \\ &= -A^0B^0 - A^0B^1 + A^2B^0 + A^2B^1,\end{aligned}$$

And so we see that $\otimes$ is not commutative.

The components of the outer product of two tensors are given by the products of the components of the

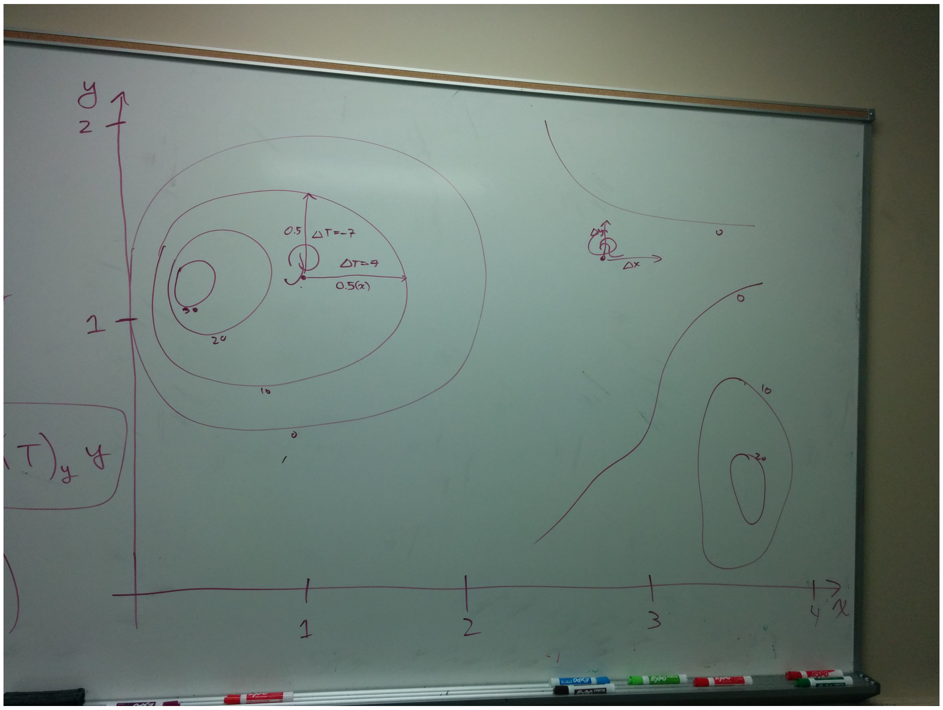


Figure 3.2: Problem 9: Isotherms.

individual tensors. Thus we can write the components as a 4×4 matrix.

$$(\tilde{p} \otimes \tilde{q})_{\alpha\beta} = p_\alpha q_\beta = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

18

(a) Find the one-forms mapped by $\mathbf{g}$ from

$$\vec{A} \underset{\mathcal{O}}{\rightarrow} (1, 0, -1, 0), \qquad \vec{B} \underset{\mathcal{O}}{\rightarrow} (0, 1, 1, 0),$$
$$\vec{C} \underset{\mathcal{O}}{\rightarrow} (-1, 0, -1, 0), \qquad \vec{D} \underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1).$$

In general,

$$\vec{V} \underset{\mathcal{O}}{\rightarrow} (V^0, V^1, V^2, V^3) \implies \tilde{V} = \mathbf{g}\vec{V} \underset{\mathcal{O}}{\rightarrow} (-V^0, V^1, V^2, V^3),$$

and so

$$\tilde{A} \underset{\mathcal{O}}{\rightarrow} (-1, 0, -1, 0), \qquad \tilde{B} \underset{\mathcal{O}}{\rightarrow} (0, 1, 1, 0),$$
$$\tilde{C} \underset{\mathcal{O}}{\rightarrow} (1, 0, -1, 0), \qquad \tilde{D} \underset{\mathcal{O}}{\rightarrow} (0, 0, 1, 1).$$

(b) Find the vectors mapped by $\mathbf{g}$ from

$$\tilde{p} \underset{\mathcal{O}}{\rightarrow} (3, 0, -1, -1), \qquad \tilde{q} \underset{\mathcal{O}}{\rightarrow} (1, -1, 1, 1),$$

$$\tilde{r} \underset{\mathcal{O}}{\rightarrow} (0, -5, -1, 0), \qquad \tilde{s} \underset{\mathcal{O}}{\rightarrow} (-2, 1, 0, 0).$$

By using the inverse tensor in reverse, we have the same effect as before, of negating the first component

$$\vec{p} \underset{\mathcal{O}}{\rightarrow} (-3, 0, -1, -1), \qquad \vec{q} \underset{\mathcal{O}}{\rightarrow} (-1, -1, 1, 1),$$

$$\vec{r} \underset{\mathcal{O}}{\rightarrow} (0, -5, -1, 0), \qquad \vec{s} \underset{\mathcal{O}}{\rightarrow} (2, 1, 0, 0).$$

20

In Euclidean 3-space, vectors and covectors are usually treated as the same, because they transform the same. We will now prove this.

(a) Show that $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} A^{\beta}$ and $P_{\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\beta}} P_{\alpha}$ are the same transformations if $\{\Lambda^{\alpha}{}_{\bar{\beta}}\}$ is equal to the transpose of its inverse.

We can write that last statement as

$$\Lambda^{\alpha}{}_{\bar{\beta}} = ((\Lambda^{\alpha}{}_{\bar{\beta}})^{-1})^T$$

and we know that

$$(\Lambda^{\alpha}{}_{\bar{\beta}})^{-1} = \Lambda^{\bar{\beta}}{}_{\alpha},$$

and also we know that the Lorentz transformation is symmetric, and so

$$(\Lambda^{\bar{\beta}}{}_{\alpha})^T = \Lambda^{\bar{\beta}}{}_{\alpha},$$

which leads us to conclude that $\Lambda^{\alpha}{}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha}$, meaning the two transformations are the same.

(b) The metric has components $\{\delta_{ij}\}$. Prove that transformations between Cartesian coordinate systems must satisfy

$$\delta_{\bar{i}\bar{j}} = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \delta_{kl},$$

and that this implies that $\Lambda^{k}{}_{\bar{i}}$ is an orthogonal matrix.

$$\delta_{\bar{i}\bar{j}} = \mathbf{g}(\vec{e}_{\bar{i}}, \vec{e}_{\bar{j}}) = \mathbf{g}(\Lambda^{k}{}_{\bar{i}} \vec{e}_k, \Lambda^{l}{}_{\bar{j}} \vec{e}_j) = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \mathbf{g}(\vec{e}_k, \vec{e}_j) = \Lambda^{k}{}_{\bar{i}} \Lambda^{l}{}_{\bar{j}} \delta_{kl}$$

Now show it is orthogonal

21

(a) A region of the t–x plane is bounded by lines $t = 0$, $t = 1$, $x = 0$, and $x = 1$. Within the plane, find the unit outward normal 1-forms and their vectors for each boundary line.

I define unit outward normals as follows:

Let $\mathcal{S}$ be a closed surface. If, for each $\vec{V}$ tangent to $\mathcal{S}$, we have $\tilde{p}(\vec{V}) = 0$, then $\tilde{p}$ is normal to $\mathcal{S}$.

In addition, if, for each $\vec{U}$ which points outwards from the surface, we have $\tilde{p}(\vec{U}) > 0$, then $\tilde{p}$ is an outward normal.

Furthermore, if $\tilde{p}^2 = \pm 1$, then it is a unit outward normal.

For the problem at hand, I define the region inside the four lines to be *Inside*, and the region outside to be *Outside*. For each of the four lines, I draw a vector $\vec{V}$ tangent (parallel) to the line, and $\vec{U}$ pointing outwards (See Figure 3.3).

It helps to look at $t = 0$ and $t = 1$ together, and likewise for x, so I will start with t. We start with an arbitrary $\tilde{p} \to_{\mathcal{O}} (p_0, p_1)$, and $\vec{V} \to_{\mathcal{O}} (0, V^1)$, where $V^1 \neq 0$.

$$\tilde{p}(\vec{V}) = p_0 \cdot 0 + p_1 V^1 = 0 \implies p_1 = 0,$$

so $\tilde{p} \to_{\mathcal{O}} (p_0, 0)$ is a normal 1-form to both lines. Now we find the corresponding *unit* normal, by taking

$$\tilde{p}^2 = \pm 1 = -(p_0)^2 \implies \tilde{p}^2 = -1 \ \& \ p_0 = \pm 1.$$

Whether we choose p_0 to be positive or negative now depends on the line we are looking at, and which direction is outward. For $t = 0$, we have a vector $\vec{U} = (-U^0, U^1)$, where $U^0 > 0$.

$$\tilde{p}(\vec{U}) = p_0(-U^0) + 0 \cdot U^1 > 0 \implies -p_0 U^0 > 0 \implies p_0 < 0,$$

so for $t = 0$ we have $\tilde{p} \to_{\mathcal{O}} (-1, 0)$, and likewise for $t = 1$ we have $\tilde{p} \to_{\mathcal{O}} (1, 0)$. To get the associated *vectors*, we apply the metric $\eta^{\alpha\beta}$, giving us $\vec{p} \to_{\mathcal{O}} (1, 0)$ for $t = 0$ and $\vec{p} \to_{\mathcal{O}} (-1, 0)$ for $t = 1$.

For $x = 0$ and $x = 1$, we instead have $\vec{V} \to_{\mathcal{O}} (V^0, 0)$, and following the same steps as before, we conclude that: for $x = 0$, $\tilde{p} \to_{\mathcal{O}} (0, -1)$, $\vec{p} \to_{\mathcal{O}} (0, -1)$, and for $x = 1$, $\tilde{p} \to_{\mathcal{O}} (0, 1)$, $\vec{p} \to_{\mathcal{O}} (0, 1)$.

Figure 3.3: Problem 21.a

(b) Let another region be bounded by the set of points $\{(1, 0), (1, 1), (2, 1)\}$. Find an outward normal for the null boundary and the associated vector.

23

(a) Prove that the set of all $\binom{M}{N}$ tensors forms a vector space, V.

Let T be the set of all $\binom{M}{N}$ tensors, $\mathbf{s}, \mathbf{p}, \mathbf{q} \in T$, $\vec{A} \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. For T to be a vector space, we must define the operations of addition, and scalar multiplication (amongst others).

Addition:

$$\mathbf{s} = \mathbf{p} + \mathbf{q} \implies \mathbf{s}(\vec{A}) = \mathbf{p}(\vec{A}) + \mathbf{q}(\vec{A})$$

Scalar Multiplication:

$$\mathbf{r} = \alpha \mathbf{p} \implies \mathbf{r}(\vec{A}) = \alpha \mathbf{p}(\vec{A})$$

(b)

Prove that a basis for T is

$$\{\vec{e}_\alpha \otimes \ldots \otimes \vec{e}_\gamma \otimes \tilde{\omega}^\mu \otimes \ldots \otimes \tilde{\omega}^\lambda\}$$

Still working on it

24 Given:

$$M^{\alpha\beta} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(a) Find:

(i)

$$M^{(\alpha\beta)} \to \begin{pmatrix} 0 & 1 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{pmatrix}; \quad M^{[\alpha\beta]} \to \begin{pmatrix} 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \frac{3}{2} \\ \frac{1}{2} & -1 & -\frac{3}{2} & 0 \end{pmatrix}$$

(ii)

$$M^{\alpha}{}_{\beta} = \eta_{\beta\mu} M^{\alpha\mu} \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iii)

$$M_{\alpha}{}^{\beta} = \eta_{\alpha\mu} M^{\mu\beta} \to \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(iv)

$$M_{\alpha\beta} = \eta_{\beta\mu} M_{\alpha}{}^{\mu} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

(b) Does it make sense to separate the $\binom{1}{1}$ tensor with components $M^{\alpha}{}_{\beta}$ into symmetric and antisymmetric parts?

No, it would not make sense. For one, the notation for (anti)symmetric tensors do not even allow one to write it sensibly ($M^{(\alpha}{}_{\beta)}$). More importantly, one argument refers to vectors, and the other to covectors, so it does not make sense to switch them.

(c)

$$\eta^{\alpha}{}_{\beta} = \eta^{\alpha\mu}\eta_{\beta\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta^{\alpha}{}_{\beta}$$

31

Still working on it

(**33**)

34 Define double-null coordinates $u = t - x$, $v = t + x$ in Minkowski space.

(a) Let $\vec{e}_u$ be the vector connecting the (u, v, y, t) coordinates $(0, 0, 0, 0)$ and $(1, 0, 0, 0)$, and let $\vec{e}_v$ be the vector connecting $(0, 0, 0, 0)$ and $(0, 1, 0, 0)$. Find $\vec{e}_u$ and $\vec{e}_v$ in terms of $\vec{e}_t$ and $\vec{e}_x$, and plot the basis vectors in a spacetime diagram of the t–x plane.

$$u = t - x = 0 \implies t = +x \qquad v = t + x = 0 \implies t = -x$$
$$u = t - x = 1 \implies t = 1 + x \qquad v = t + x = 1 \implies t = 1 - x$$

We draw the vectors $\vec{e}_u$ and $\vec{e}_v$ in Figure 3.4, such that they point from the appropriate points of intersection on these lines of constant u and v. From this it is obvious that $\vec{e}_v + \vec{e}_u = \vec{e}_t$, and that $\vec{e}_v - \vec{e}_u = \vec{e}_x$, or likewise $\vec{e}_v = \vec{e}_t - \vec{e}_u$ and $\vec{e}_u = \vec{e}_v - \vec{e}_x$. This is a system of 2 equations with two unknowns.

$$\vec{e}_v = \vec{e}_t - \vec{e}_v + \vec{e}_x \implies \qquad \vec{e}_v = \frac{1}{2}(\vec{e}_t + \vec{e}_x),$$
$$\vec{e}_u = \frac{1}{2}(\vec{e}_t + \vec{e}_x) - \vec{e}_x \implies \qquad \vec{e}_u = \frac{1}{2}(\vec{e}_t - \vec{e}_x).$$

(b) Show that $\vec{e}_\alpha$, $\alpha \in \{u, v, y, z\}$ form a basis for vectors in Minkowski space.

$$\begin{aligned} \vec{A} = A^\alpha \vec{e}_\alpha &= A^u \vec{e}_u + A^v \vec{e}_v + A^y \vec{e}_y + A^z \vec{e}_z \\ &= \frac{A^u}{2}(\vec{e}_t - \vec{e}_x) + \frac{A^v}{2}(\vec{e}_t + \vec{e}_x) + A^y \vec{e}_y + A^z \vec{e}_z \\ &= \frac{1}{2}(A^v + A^u)\vec{e}_t + \frac{1}{2}(A^v - A^u)\vec{e}_x + A^y \vec{e}_y + A^z \vec{e}_z \end{aligned}$$

If we let $A^t = \frac{1}{2}(A^v + A^u)$ and $A^x = \frac{1}{2}(A^v - A^u)$, then

$$\vec{A} = A^\alpha \vec{e}_\alpha = A^t \vec{e}_t + A^x \vec{e}_x + A^y \vec{e}_y + A^z \vec{e}_z$$

(c) Find the components of the metric tensor, $\mathbf{g}$ in this new basis.

To make this concise, we will begin with some definitions. Let $w \in \{u, v\}$, and $q \in \{y, z\}$. We also define

$$\lambda(w) \equiv \begin{cases} -1, & \text{if } w = u, \\ +1, & \text{if } w = v. \end{cases}$$

It follows that

$$\vec{e}_w = \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x).$$

Now we can show that

$$\begin{aligned} g_{ww} = \vec{e}_w \cdot \vec{e}_w &= \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 2\lambda(\vec{e}_t \cdot \vec{e}_x) + \lambda^2(\vec{e}_x \cdot \vec{e}_x)] \\ &= \frac{1}{4}(-1 + 2\lambda \cdot 0 + 1 \cdot 1) = 0, \end{aligned}$$

so $g_{uu} = g_{vv} = 0$.

For the u and v cross terms, we have

$$\begin{aligned} g_{uv} = g_{vu} = \vec{e}_u \cdot \vec{e}_v &= \frac{1}{2}(\vec{e}_t - \vec{e}_x) \cdot \frac{1}{2}(\vec{e}_t + \vec{e}_x) \\ &= \frac{1}{4}[\vec{e}_t \cdot \vec{e}_t + 0 \cdot \vec{e}_t \cdot \vec{e}_x - \vec{e}_x \cdot \vec{e}_x] \\ &= \frac{1}{4}(-1 + 0 - 1) = -\frac{1}{2} \end{aligned}$$

For the w with y and z cross terms we have

$$\begin{aligned} g_{wq} = \vec{e}_w \cdot \vec{e}_q &= \frac{1}{2}(\vec{e}_t + \lambda \vec{e}_x) \cdot \vec{e}_q \\ &= \frac{1}{2}[\vec{e}_t \cdot \vec{e}_t + \lambda \vec{e}_x \cdot \vec{e}_x] \\ &= 0 \end{aligned}$$

so $g_{uy} = g_{vy} = g_{uz} = g_{vz} = 0$. We also already know $g_{yy} = g_{zz} = 1$, and $g_{yz} = g_{zy} = 0$, so we can write the components of the metric tensor in this new coordinate system as

$$g_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) Show that $\vec{e}_u$ and $\vec{e}_v$ are null, but not orthogonal.

$$\vec{e}_u \cdot \vec{e}_u = g_{uu} = 0 \implies \vec{e}_u \text{ is null}$$

$$\vec{e}_v \cdot \vec{e}_v = g_{vv} = 0 \implies \vec{e}_v \text{ is null}$$

$$\vec{e}_u \cdot \vec{e}_v = g_{uv} = -\frac{1}{2} \neq 0 \implies \vec{e}_u \text{ and } \vec{e}_v \text{ are not orthogonal.}$$

(e) Compute the four one-forms $\tilde{\mathrm{d}}u$, $\tilde{\mathrm{d}}v$, $\mathbf{g}(\vec{e}_u,)$, and $\mathbf{g}(\vec{e}_v,)$ in terms of $\tilde{\mathrm{d}}t$ and $\tilde{\mathrm{d}}x$.

$$\tilde{\mathrm{d}}\phi \to_{\mathcal{O}} \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right),$$

so

$$\tilde{\mathrm{d}}t \to_{\mathcal{O}} (1,0,0,0), \qquad \tilde{\mathrm{d}}x \to_{\mathcal{O}} (0,1,0,0),$$
$$\tilde{\mathrm{d}}u \to_{\mathcal{O}} \frac{1}{2}(1,-1,0,0), \qquad \tilde{\mathrm{d}}u \to_{\mathcal{O}} \frac{1}{2}(1,1,0,0),$$

from which it is obvious that

$$\tilde{\mathrm{d}}u = \frac{1}{2}(\tilde{\mathrm{d}}t - \tilde{\mathrm{d}}x), \qquad \tilde{\mathrm{d}}v = \frac{1}{2}(\tilde{\mathrm{d}}t + \tilde{\mathrm{d}}x).$$

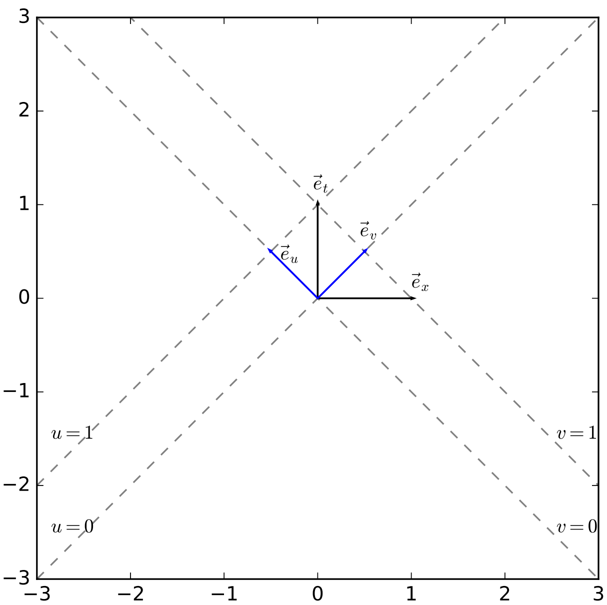


Figure 3.4: Problem 34a: Spacetime diagram of double-null coordinate basis vectors in t–x plane.

Chapter 5

Preface to Curvature

5.8 Exercises

1

(a) Repeat the argument leading to Equation 5.1, but this time assume that only a fraction $\epsilon < 1$ of the mass's kinetic energy is converted into a photon.

If only a fraction ϵ of the energy is converted into a photon, then it will start with an energy of $\epsilon(m + mgh + \mathcal{O}(\boldsymbol{v}^4)$, but once it reaches the top it should have an energy of ϵm, as it loses the component due to gravitational potential energy. Thus

$$\frac{E'}{E} = \frac{\epsilon m}{\epsilon(m + mgh + \mathcal{O}(\boldsymbol{v}^4))} = \frac{m}{m + mgh + \mathcal{O}(\boldsymbol{v}^4)} = 1 - gh + \mathcal{O}\Big(\boldsymbol{v}^4\Big)$$

(b) Assume Equation 5.1 does not hold. Devise a perpetual motion device.

If we assume that the photon does not return to an energy m once it reaches the top, but instead has an energy $m' > m$, then we could create the perpetual motion device shown in Figure 5.1. A black box consumes the photon with energy m', and splits it into a new object of mass m, and a photon of energy $m' - m$. The object repeats the action of the original falling mass, creating an infinite loop.

2 Explain why a uniform gravitational field would not be able to create tides on Earth.

Tides depend on there being a gravitational field gradient. If the curvature closer to the source of the field (e.g. the Moon) is greater than it is further away, then the closer side will move towards the source more than the further side, thus creating tides. In the absense of such a gradient, there would be no difference in curvature between the two sides, and thus they would not stretch relative to each other.

7 Calculate the components of $\Lambda^{\alpha'}{}_{\beta}$ and $\Lambda^{\mu}{}_{\nu'}$ for transformations $(x, y) \leftrightarrow (r, \theta)$.

$$\begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix}$$

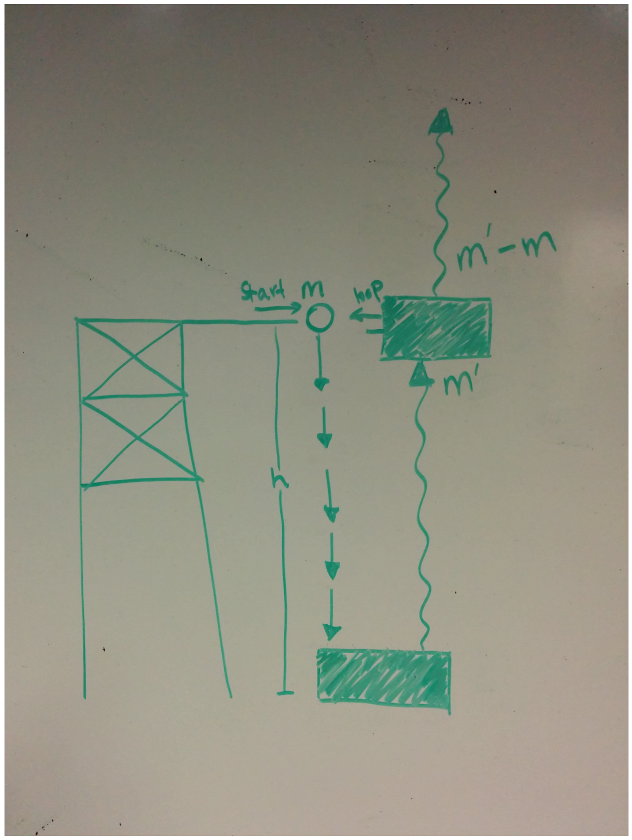


Figure 5.1: Problem 1: Perpetual motion device.

$$= \begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad = \begin{pmatrix} x/\sqrt{x^2+y^2} & -y \\ y/\sqrt{x^2+y^2} & x \end{pmatrix} \begin{pmatrix} \Delta r \\ \Delta\theta \end{pmatrix}$$

$$\Lambda^r{}_x = x/\sqrt{x^2+y^2} = \cos\theta \qquad \Lambda^x{}_r = \cos\theta = x/\sqrt{x^2+y^2}$$

$$\Lambda^r{}_y = y/\sqrt{x^2+y^2} = \sin\theta \qquad \Lambda^y{}_r = \sin\theta = y/\sqrt{x^2+y^2}$$

$$\Lambda^\theta{}_x = -y/(x^2+y^2) = -(1/r)\sin\theta \qquad \Lambda^x{}_\theta = -r\sin\theta = -y$$

$$\Lambda^\theta{}_y = x/(x^2+y^2) = (1/r)\cos\theta \qquad \Lambda^y{}_\theta = r\cos\theta = x$$

8

(a) $f \equiv x^2+y^2+2xy$, $\vec{V} \underset{(x,y)}{\rightarrow} (x^2+3y, y^2+3x)$, $\vec{W} \underset{(r,\theta)}{\rightarrow} (1,1)$. Express $f = f(r,\theta)$, and find the components of $\vec{V}$ and $\vec{W}$ in a polar basis, as functions of r and θ.

$$f = x^2+y^2+2xy = (x+y)^2$$

$$= (r\cos\theta + r\sin\theta)^2 = r^2\sin^2\theta + r^2\cos^2\theta + 2r^2\sin\theta\cos\theta$$

$$= r^2(1+\sin(2\theta))$$

$$\vec{V} \underset{(x,y)}{\to} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix}$$

$$\vec{V} \underset{(r,\theta)}{\to} \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix}$$

$$\underset{(r,\theta)}{\to} \begin{pmatrix} r^2\cos^2\theta + 6r\sin\theta\cos\theta + r^2\sin^3\theta \\ -r\cos^2\theta\sin\theta - 3\sin^2\theta + r\sin^2\theta\cos\theta + 3\cos^2\theta \end{pmatrix}$$

$$\underset{(r,\theta)}{\to} \begin{pmatrix} r^2(\sin^3\theta + \cos^3\theta) + 6r\sin\theta\cos\theta \\ r\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix}$$

$$\underset{(r,\theta)}{\to} \begin{pmatrix} r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta) \\ (r/2)\sin(2\theta)(\sin\theta - \cos\theta) + 3\cos(2\theta) \end{pmatrix}$$

$$\vec{W} \underset{(r,\theta)}{\to} \begin{pmatrix} \cos\theta & \sin\theta \\ -(1/r)\sin\theta & (1/r)\cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underset{(r,\theta)}{\to} \begin{pmatrix} \cos\theta + \sin\theta \\ (1/r)(\cos\theta - \sin\theta) \end{pmatrix}$$

(b) Express the components of $\tilde{\mathrm{d}}f$ in (x,y) and obtain them in (r,θ) by:

(i) using direct calculation in (r,θ):

$$\tilde{\mathrm{d}}f \underset{(r,\theta)}{\to} \big(\partial f/\partial r\ ,\ \partial f/\partial\theta\big) = \Big(2r(1+\sin(2\theta)), 2r^2\cos(2\theta)\Big)$$

(ii) transforming the components in (x,y):

$$\tilde{\mathrm{d}}f \underset{(x,y)}{\to} \big(\partial f/\partial x\ ,\ \partial f/\partial y\big) = \big(2(x+y), 2(x+y)\big) = \big(2r(\cos\theta + \sin\theta), 2r(\cos\theta + \sin\theta)\big)$$

$$\Big((\tilde{\mathrm{d}}f)_r \quad (\tilde{\mathrm{d}}f)_\theta\Big) = \Big(1 \quad 1\Big) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \big[2r(\cos\theta + \sin\theta)\big]$$

$$= \Big(2r(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta) \quad 2r^2(\cos^2\theta - \sin^2\theta)\Big)$$

$$= \Big(2r(1+\sin(2\theta)) \quad 2r^2\cos(2\theta)\Big)$$

(c) Now find the (r,θ) components of the one-forms $\tilde{V}$ and $\tilde{W}$ associated with the vectors $\vec{V}$ and $\vec{W}$ by

(i) using the metric tensor in (r,θ):

$$
\begin{aligned}
V_r &= g_{r\alpha}V^\alpha = g_{rr}V^r + g_{r\theta}V^\theta \\
&= r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta) \\
V_\theta &= g_{\theta r}V^r + g_{\theta\theta}V^\theta = (1/2)r^3\sin(2\theta)(\sin\theta - \cos\theta) + 3r^2\cos(2\theta) \\
W_r &= g_{r\alpha}W^\alpha = g_{rx}W^x + g_{ry}W^y \\
&= 1(\cos\theta + \sin\theta) + 0\big[(1/r)(\cos\theta - \sin\theta)\big] \\
&= \cos\theta + \sin\theta \\
W_\theta &= g_{\theta x}W^x + g_{\theta y}W^y = \\
&= 0(\cos\theta + \sin\theta) + r^2\big[r(\cos\theta - \sin\theta)\big] \\
&= r(\cos\theta - \sin\theta)
\end{aligned}
$$

(ii) using the metric tensor in (x, y) and then doing a coordinate transformation:

$$
\begin{aligned}
V_x &= V^x; \quad V_y = V^y \\
V_r &= \Lambda^\alpha{}_r V_\alpha = \Lambda^x{}_r V_x + \Lambda^y{}_r V_y \\
&= \cos\theta V_x + \sin\theta V_y \\
&= r^2\cos^3\theta + (3/2)r\sin(2\theta) + r^2\sin^3\theta + (3/2)r\sin(2\theta) \\
&= r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta) \\
V_\theta &= \Lambda^\alpha{}_\theta V_\alpha = \Lambda^x{}_\theta V_x + \Lambda^y{}_\theta V_y \\
&= (-r\sin\theta)V_x + (r\cos\theta)V_y \\
&= -r^3\cos^2\theta\sin\theta - 3r^2\sin^2\theta + r^3\sin^2\theta\cos\theta + 3r^2\cos^2\theta \\
&= r^3\sin\theta\cos\theta(\sin\theta - \cos\theta) + 3r^2(\cos^2\theta - \sin^2\theta) \\
&= (1/2)r^3\sin(2\theta)(\sin\theta - \cos\theta) + 3r^2\cos(2\theta) \\
W_x &= W^x = W_y = W^y = 1 \\
W_r &= \Lambda^\alpha{}_r W_\alpha = \Lambda^x{}_r W_x + \Lambda^y{}_r W_y \\
&= \cos\theta + \sin\theta \\
W_\theta &= \Lambda^\alpha{}_\theta W_\alpha = \Lambda^x{}_\theta W_x + \Lambda^y{}_\theta W_y \\
&= -r\sin\theta + r\cos\theta \\
&= r(\cos\theta - \sin\theta)
\end{aligned}
$$

11 Consider $V \underset{(x,y)}{\rightarrow} (x^2 + 3y, y^2 + 3x)$.

(a) Find $V^{\alpha}{}_{,\beta}$ in Cartesian coordinates.

$$V^x{}_{,x} = 2x; \quad V^y{}_{,y} = 2y; \quad V^x{}_{,y} = V^y{}_{,x} = 3.$$

(b)

$$\begin{aligned}
V^{\mu'}{}_{;\nu'} &= \Lambda^{\mu'}{}_{\alpha}\Lambda^{\beta}{}_{\nu'}V^{\alpha}{}_{,\beta} \\
V^r{}_{;r} &= \Lambda^r{}_x\Lambda^x{}_rV^x{}_{,x} + \Lambda^r{}_y\Lambda^y{}_rV^y{}_{,y} + \Lambda^r{}_x\Lambda^y{}_rV^x{}_{,y} + \Lambda^r{}_y\Lambda^x{}_rV^y{}_{,x} \\
&= (\cos^2\theta)(2r\cos\theta) + (\sin^2\theta)(2r\sin\theta) + (\sin\theta\cos\theta)(3) + (\sin\theta\cos\theta)(3) \\
&= 2r(\cos^3\theta + \sin^3\theta) + 3\sin(2\theta) \\
V^\theta{}_{;\theta} &= \Lambda^\theta{}_x\Lambda^\theta{}_rV^x{}_{,x} + \Lambda^\theta{}_y\Lambda^\theta{}_rV^y{}_{,y} + \Lambda^\theta{}_x\Lambda^y{}_\theta V^x{}_{,y} + \Lambda^\theta{}_y\Lambda^\theta{}_rV^y{}_{,x} \\
&= (\sin^2\theta)(2r\cos\theta) + (\cos^2\theta)(2r\sin\theta) + (-\sin\theta\cos\theta)(3) + (-\sin\theta\cos\theta)(3) \\
&= \sin(2\theta)[r(\sin\theta + \cos\theta) - 3] \\
V^r{}_{;\theta} &= \Lambda^r{}_x\Lambda^x{}_\theta V^x{}_{,x} + \Lambda^r{}_y\Lambda^y{}_\theta V^y{}_{,y} + \Lambda^r{}_x\Lambda^y{}_\theta V^x{}_{,y} + \Lambda^r{}_y\Lambda^x{}_\theta V^y{}_{,x} \\
&= (-r\sin\theta\cos\theta)(2r\cos\theta) + (r\sin\theta\cos\theta)(2r\sin\theta) + (r\cos^2\theta)(3) + (-r\sin^2\theta) \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta) \\
V^\theta{}_{;r} &= \Lambda^\theta{}_x\Lambda^x{}_rV^x{}_{,x} + \Lambda^\theta{}_y\Lambda^y{}_rV^y{}_{,y} + \Lambda^\theta{}_x\Lambda^y{}_rV^x{}_{,y} + \Lambda^\theta{}_y\Lambda^x{}_rV^y{}_{,x} \\
&= (-(1/r)\sin\theta\cos\theta)(2r\cos\theta) + ((1/r)\sin\theta\cos\theta)(2r\sin\theta) + (-(1/r)\sin^2\theta)(3) + ((1/r)\cos^2\theta)(3) \\
&= \sin(2\theta)(\sin\theta - \cos\theta) + \frac{3}{r}\cos(2\theta)
\end{aligned}$$

(c) compute $V^{\mu'}{}_{;\nu'}$ directly in polars using the Christoffel symbols.
Recall that we have $\Gamma^{\mu}{}_{rr} = \Gamma^r{}_{r\theta} = \Gamma^\theta{}_{\theta\theta} = 0$, $\Gamma^\theta{}_{r\theta} = 1/r$, and $\Gamma^r{}_{\theta\theta} = -r$.

$$\begin{aligned}
V^{\mu'}{}_{;\nu'} &= V^{\mu'}{}_{,\nu'} + V^{\alpha'}\Gamma^{\mu'}{}_{\alpha'\nu'} \\
V^r{}_{;r} &= V^r{}_{,r} + V^{\alpha'}\Gamma^r{}_{\alpha' r} \\
V^r{}_{,r} &= \partial V^r/\partial r = 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
V^\alpha\Gamma^r{}_{\alpha r} &= V^r\Gamma^r{}_{rr} + V^\theta\Gamma^r{}_{\theta r} = 0 \\
V^r{}_{;r} &= V^r{}_{,r} = 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
V^\theta{}_{;\theta} &= V^\theta{}_{,\theta} + V^{\alpha'}\Gamma^\theta{}_{\alpha'\theta} \\
V^\theta{}_{,\theta} &= \partial V^\theta \big/ \partial\theta = (r/2)\sin(2\theta)(\sin\theta + \cos\theta) + r\cos(2\theta)(\sin\theta - \cos\theta) - 6\sin(2\theta) \\
V^{\alpha'}\Gamma^\theta{}_{\alpha'\theta} &= V^r\Gamma^\theta{}_{r\theta} + V^\theta\Gamma^\theta{}_{\theta\theta} \\
&= [r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta)](1/r) \\
&= r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta)
\end{aligned}$$

$$
\begin{aligned}
V^{\theta}{}_{;\theta} &= \sin(2\theta)[r(\sin\theta + \cos\theta) - 3] \\
V^{r}{}_{;\theta} &= V^{r}{}_{,\theta} + V^{r}\Gamma^{r}{}_{r\theta} + V^{\theta}\Gamma^{r}{}_{\theta\theta} \\
&= V^{r}{}_{,\theta} + V^{\theta}\Gamma^{r}{}_{\theta\theta} = \partial V^{r}/\partial\theta - rV^{\theta} \\
&= 6r\cos(2\theta) + (3/2)r^2\sin(2\theta)(\sin\theta - \cos\theta) - ((1/2)r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta)) \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta) \\
V^{\theta}{}_{;r} &= V^{\theta}{}_{,r} + V^{r}\Gamma^{\theta}{}_{rr} + V^{\theta}\Gamma^{\theta}{}_{\theta r} = V^{\theta}{}_{,r} + \frac{1}{r}V^{\theta} \\
&= (1/2)\sin(2\theta)(\sin\theta - \cos\theta) + (1/2)\sin(2\theta)(\sin\theta - \cos\theta) + (3/r)\cos(2\theta) \\
&= \sin(2\theta)(\sin\theta - \cos\theta) + (3/r)\cos(2\theta)
\end{aligned}
$$

(d) Calculate the divergence using the results from part (a)

$$V^{\alpha}{}_{,\alpha} = V^{x}{}_{,x} + V^{y}{}_{,y} = 2(x + y) = 2r(\sin\theta + \cos\theta)$$

(e) Calculate the divergence using the results from either part (b) or (c).

$$
\begin{aligned}
V^{\mu'}{}_{;\mu'} &= V^{r}{}_{;r} + V^{\theta}{}_{;\theta} \\
&= 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) + \sin(2\theta)[r(\sin\theta + \cos\theta) - 3] \\
&= 2r(\sin\theta + \cos\theta)
\end{aligned}
$$

(f) Compute $V^{\mu'}{}_{;\mu'}$ using Equation 5.56.

$$V^{\mu'}{}_{;\mu'} = \frac{1}{r}\frac{\partial}{\partial r}(rV^{r}) + \frac{\partial}{\partial\theta}(V^{\theta}) = 2r(\sin\theta + \cos\theta)$$

12

$$\tilde{p} \underset{(x,y)}{\rightarrow} (x^2 + 3y, y^2 + 3x).$$

(a) Find the components $p_{\alpha,\beta}$ in Cartesian coordinates.

Since $p_{\alpha,\beta} = \partial p_{\alpha}/\partial x^{\beta}$, it's simply $p_{x,x} = 2x$, $p_{y,y} = 2y$, and $p_{x,y} = p_{y,x} = 3$.

(b) Find the components $p_{\mu';\nu'}$ in polar coordinates by using the transformation $\Lambda^{\alpha}{}_{\mu'}\Lambda^{\beta}{}_{\nu'}p_{\alpha,\beta}$.

$$
\begin{aligned}
p_{r;r} &= (\Lambda^{x}{}_{r})^2 p_{x,x} + (\Lambda^{y}{}_{r})^2 p_{y,y} + 2\Lambda^{x}{}_{r}\Lambda^{y}{}_{r}p_{x,y} \\
&= (\cos^2\theta)(2r\cos\theta) + (\sin^2\theta)(2r\sin\theta) + 2(\sin\theta\cos\theta)(3) \\
&= 2r(\sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
p_{\theta;\theta} &= (\Lambda^{x}{}_{\theta})^2 p_{x,x} + \left(\Lambda^{y}{}_{\theta}\right)^2 p_{y,y} + 2\Lambda^{x}{}_{\theta}\Lambda^{y}{}_{\theta}p_{x,y} \\
&= (-r\sin\theta)^2(2r\cos\theta) + (r\cos\theta)^2(2r\sin\theta) + 2(3(-r\sin\theta)(r\cos\theta)) \\
&= r^2\sin(2\theta)(r(\sin\theta + \cos\theta) - 3)
\end{aligned}
$$

$$\begin{aligned}
p_{r;\theta} &= \Lambda^x{}_r \Lambda^x{}_\theta p_{x,x} + \Lambda^y{}_r \Lambda^y{}_\theta p_{y,y} + \Lambda^x{}_r \Lambda^y{}_\theta p_{x,y} + \Lambda^y{}_r \Lambda^x{}_\theta p_{y,x} \\
&= (-r\sin\theta\cos\theta)(2r\cos\theta) + (r\sin\theta\cos\theta)(2r\sin\theta) + 3(r\cos^2\theta - r\sin^2\theta) \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta),
\end{aligned}$$

and by the symmetry of $p_{\alpha,\beta}$ in Cartesian coordinates, $p_{\theta;r} = p_{r;\theta}$.

(c) Now find $p_{\mu';\nu'}$ using the Christoffel symbols.

$$\begin{aligned}
p_{r;r} &= p_{r,r} - p_r\Gamma^r{}_{rr} - p_\theta\Gamma^\theta{}_{rr} = p_{r,r} = \partial p_r/\partial r \\
&= \partial/\partial r \left[r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta)\right] = 2r(sin^3\theta + \cos^3\theta) + 3\sin(2\theta) \\
p_{\theta;\theta} &= p_{\theta,\theta} - p_r\Gamma^r{}_{\theta\theta} - p_\theta\Gamma^\theta{}_{\theta\theta} = p_{\theta,\theta} + rp_r = \partial p_\theta/\partial\theta \\
&= \partial/\partial\theta \left[(1/2)r^3\sin(2\theta)(\sin\theta - \cos\theta) + 3r^2\cos(2\theta)\right] + r\left[r^2(\cos^3\theta + \sin^3\theta) + 3r\sin(2\theta)\right] \\
&= r^2\sin(2\theta)\left[r(\sin\theta + \cos\theta) - 3\right] \\
p_{r;\theta} &= p_{r,\theta} - p_r\Gamma^r{}_{r\theta} - p_\theta\Gamma^\theta{}_{r\theta} = \partial p_r/\partial\theta - (1/r)p_\theta \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta) \\
p_{\theta;r} &= p_{\theta,r} - p_r\Gamma^r{}_{\theta r} - p_\theta\Gamma^\theta{}_{\theta r} = \partial p_\theta/\partial r - (1/r)p_\theta \\
&= r^2\sin(2\theta)(\sin\theta - \cos\theta) + 3r\cos(2\theta)
\end{aligned}$$

13 Show in polars that $g_{\mu'\alpha'}V^{\alpha'}{}_{;\nu'} = p_{\mu';\nu'}$.

$$\begin{aligned}
g_{r\alpha'}V^{\alpha'}{}_{;r} &= g_{rr}V^r{}_{;r} + g_{r\theta}V^\theta{}_{;r} \\
&= 1V^r{}_{;r} = p_{r;r} \\
g_{\theta\alpha'}V^{\alpha'}{}_{;\theta} &= g_{\theta r}V^r{}_{;\theta} + g_{\theta\theta}V^\theta{}_{;\theta} \\
&= r^2V^\theta{}_{;\theta} = p_{\theta;\theta} \\
g_{r\alpha'}V^{\alpha'}{}_{;\theta} &= g_{rr}V^r{}_{;\theta} + g_{r\theta}V^\theta{}_{;\theta} \\
&= 1V^r{}_{;\theta} = p_{\theta;r} \\
g_{\theta\alpha'}V^{\alpha'}{}_{;r} &= g_{\theta r}V^r{}_{;r} + g_{\theta\theta}V^\theta{}_{;r} \\
&= r^2V^\theta{}_{;r} = p_{\theta;r}
\end{aligned}$$

14 Compute $\boldsymbol{\nabla}_\beta A^{\mu\nu}$ for the tensor $\boldsymbol{A}$ with components:

$$A^{rr} = r^2, \qquad A^{r\theta} = r\sin\theta,$$

$$A^{\theta\theta} = \tan\theta, \qquad A^{\theta r} = r\cos\theta$$

$$A^{rr}{}_{,r} = 2r \qquad A^{rr}{}_{,\theta} = 0$$
$$A^{\theta\theta}{}_{,r} = 0 \qquad A^{\theta\theta}{}_{,\theta} = \sec^2\theta$$
$$A^{r\theta}{}_{,r} = \sin\theta \qquad A^{r\theta}{}_{,\theta} = r\cos\theta$$
$$A^{\theta r}{}_{,r} = \cos\theta \qquad A^{\theta r}{}_{,\theta} = -r\sin\theta$$

$$\boldsymbol{\nabla}_\beta A^{\mu\nu} = A^{\mu\nu}{}_{,\beta} + A^{\alpha\nu}\Gamma^{\mu}{}_{\alpha\beta} + A^{\mu\alpha}\Gamma^{\nu}{}_{\alpha\beta}$$

$$\begin{aligned}
\boldsymbol{\nabla}_r A^{rr} &= A^{rr}{}_{,r} + A^{\alpha r}\Gamma^{r}{}_{\alpha r} + A^{r\alpha}\Gamma^{r}{}_{\alpha r}\\
&= A^{rr}{}_{,r} + A^{rr}\Gamma^{r}{}_{rr} + A^{\theta r}\Gamma^{r}{}_{\theta r} + A^{rr}\Gamma^{r}{}_{rr} + A^{r\theta}\Gamma^{r}{}_{\theta r}\\
&= A^{rr}{}_{,r} = 2r\\
\boldsymbol{\nabla}_\theta A^{rr} &= A^{rr}{}_{,\theta} + A^{\alpha r}\Gamma^{r}{}_{\alpha\theta} + A^{r\alpha}\Gamma^{r}{}_{\alpha\theta}\\
&= A^{rr}{}_{,\theta} + A^{rr}\Gamma^{r}{}_{r\theta} + A^{\theta r}\Gamma^{r}{}_{\theta\theta} + A^{rr}\Gamma^{r}{}_{r\theta} + A^{r\theta}\Gamma^{r}{}_{\theta\theta}\\
&= (A^{\theta r} + A^{r\theta})\Gamma^{r}{}_{\theta\theta} = -r^2(\sin\theta + \cos\theta)\\
\boldsymbol{\nabla}_r A^{\theta\theta} &= A^{\theta\theta}{}_{,r} + A^{\alpha\theta}\Gamma^{\theta}{}_{\alpha r} + A^{\theta\alpha}\Gamma^{\theta}{}_{\alpha r}\\
&= A^{\theta\theta}{}_{,r} + A^{r\theta}\Gamma^{\theta}{}_{rr} + A^{\theta\theta}\Gamma^{\theta}{}_{\theta r} + A^{\theta r}\Gamma^{\theta}{}_{rr} + A^{\theta\theta}\Gamma^{\theta}{}_{\theta r}\\
&= 2A^{\theta\theta}\Gamma^{\theta}{}_{\theta r} = (2/r)\tan\theta\\
\boldsymbol{\nabla}_\theta A^{\theta\theta} &= A^{\theta\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^{\theta}{}_{\alpha\theta} + A^{\theta\alpha}\Gamma^{\theta}{}_{\alpha\theta}\\
&= A^{\theta\theta}{}_{,\theta} + A^{r\theta}\Gamma^{\theta}{}_{r\theta} + A^{\theta\theta}\Gamma^{\theta}{}_{\theta\theta} + A^{\theta r}\Gamma^{\theta}{}_{r\theta} + A^{\theta\theta}\Gamma^{\theta}{}_{\theta\theta}\\
&= A^{\theta\theta}{}_{,\theta} + (A^{r\theta} + A^{\theta r})\Gamma^{\theta}{}_{r\theta} = \sin\theta + \cos\theta + \sec^2\theta\\
\boldsymbol{\nabla}_r A^{r\theta} &= A^{r\theta}{}_{,r} + A^{\alpha\theta}\Gamma^{r}{}_{\alpha r} + A^{r\alpha}\Gamma^{\theta}{}_{\alpha r}\\
&= A^{r\theta}{}_{,r} + A^{r\theta}\Gamma^{r}{}_{rr} + A^{\theta\theta}\Gamma^{r}{}_{\theta r} + A^{rr}\Gamma^{\theta}{}_{rr} + A^{r\theta}\Gamma^{\theta}{}_{\theta r}\\
&= A^{r\theta}{}_{,r} + A^{r\theta}\Gamma^{\theta}{}_{\theta r} = 2\sin\theta\\
\boldsymbol{\nabla}_\theta A^{r\theta} &= A^{r\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^{r}{}_{\alpha\theta} + A^{r\alpha}\Gamma^{\theta}{}_{\alpha\theta}\\
&= A^{r\theta}{}_{,\theta} + A^{r\theta}\Gamma^{r}{}_{r\theta} + A^{\theta\theta}\Gamma^{r}{}_{\theta\theta} + A^{rr}\Gamma^{\theta}{}_{r\theta} + A^{r\theta}\Gamma^{\theta}{}_{\theta\theta}\\
&= A^{r\theta}{}_{,\theta} + A^{\theta\theta}\Gamma^{r}{}_{\theta\theta} + A^{rr}\Gamma^{\theta}{}_{r\theta} = r(1 + \cos\theta - \tan\theta)\\
\boldsymbol{\nabla}_r A^{\theta r} &= A^{\theta r}{}_{,r} + A^{\alpha r}\Gamma^{\theta}{}_{\alpha r} + A^{\theta\alpha}\Gamma^{r}{}_{\alpha r}\\
&= A^{\theta r}{}_{,r} + A^{rr}\Gamma^{\theta}{}_{rr} + A^{\theta r}\Gamma^{\theta}{}_{\theta r} + A^{\theta r}\Gamma^{r}{}_{rr} + A^{\theta\theta}\Gamma^{r}{}_{\theta r}\\
&= A^{\theta r}{}_{,r} + A^{\theta r}\Gamma^{\theta}{}_{\theta r} = 2\cos\theta\\
\boldsymbol{\nabla}_\theta A^{\theta r} &= A^{\theta r}{}_{,\theta} + A^{\alpha r}\Gamma^{\theta}{}_{\alpha\theta} + A^{\theta\alpha}\Gamma^{r}{}_{\alpha\theta}\\
&= A^{\theta r}{}_{,\theta} + A^{rr}\Gamma^{\theta}{}_{r\theta} + A^{\theta r}\Gamma^{\theta}{}_{\theta\theta} + A^{\theta r}\Gamma^{r}{}_{r\theta} + A^{\theta\theta}\Gamma^{r}{}_{\theta\theta}\\
&= A^{\theta r}{}_{,\theta} + A^{rr}\Gamma^{\theta}{}_{r\theta} + A^{\theta\theta}\Gamma^{r}{}_{\theta\theta} = -r\sin\theta
\end{aligned}$$

15 Find the components of $V^{\alpha}{}_{;\beta;\gamma}$ for the vector $V^r = 1$, $V^\theta = 0$.

We start by finding the components of $V^{\alpha}{}_{;\beta}$.

$$V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + V^{\mu}\Gamma^{\alpha}{}_{\mu\beta}.$$

By noting that $V^{\alpha}{}_{,\beta} = V^{\theta} = \Gamma^{r}{}_{rr} = \Gamma^{r}{}_{r\theta} = 0$, we can simplify this to

$$V^{\alpha}{}_{;\beta} = V^{r}\Gamma^{\alpha}{}_{r\beta},$$

which means

$$V^{r}{}_{;r} = V^{r}{}_{;\theta} = V^{\theta}{}_{;r} = 0; \quad V^{\theta}{}_{;\theta} = \frac{1}{r}.$$

Now we can say

$$V^{\alpha}{}_{;\beta;\mu} = \boldsymbol{\nabla}_{\mu}V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{;\beta,\mu} + V^{\gamma}{}_{;\beta}\,\Gamma^{\alpha}{}_{\gamma\mu} - V^{\alpha}{}_{;\gamma}\,\Gamma^{\gamma}{}_{\beta\mu}.$$

Note that $V^{\theta}{}_{;\theta}$ is a function only of r, and so $V^{\theta}{}_{;\theta,r} = -1/r^2$, and all other partial derivatives are zero. We can also see by inspecting the components, that $V^{r}{}_{;\mu;\nu} = V^{\theta}{}_{;\mu}\,\Gamma^{r}{}_{\theta\mu}$, as all other components go to zero. Likewise, we can see that $V^{\theta}{}_{;r;\mu} = -V^{\theta}{}_{;\theta}\,\Gamma^{\theta}{}_{r\mu}$. It then becomes easy to find all the individual components. I summarize their values in Table 5.1.

16 Repeat the steps leading from Equation 5.74 to 5.75.

Recalling that $g_{\alpha\mu;\beta} = 0$, we can rewrite Equation 5.72 as

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}{}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu}.$$

Now if we switch the β and μ indices, and then switch the α and β indices, we get two more equations,

$$g_{\alpha\mu,\beta} = \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta}g_{\alpha\nu},$$
$$g_{\beta\mu,\alpha} = \Gamma^{\nu}{}_{\beta\alpha}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\alpha}g_{\beta\nu}.$$

Now we add the first two equations and subtract the third, getting

$$\begin{aligned} g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \Gamma^{\nu}{}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta}g_{\alpha\nu} - \Gamma^{\nu}{}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}{}_{\mu\alpha}g_{\beta\nu} \\ &= \Gamma^{\nu}{}_{\alpha\mu}g_{\beta\nu} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} - \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} - \Gamma^{\nu}{}_{\alpha\mu}g_{\beta\nu} \end{aligned}$$

α	β	μ	$V^{\alpha}{}_{;\beta;\mu}$
θ	θ	θ	0
θ	θ	r	$-1/r^2$
θ	r	θ	$-1/r^2$
θ	r	r	0
r	θ	θ	-1
r	θ	r	0
r	r	θ	0
r	r	r	0

Table 5.1: Components of the tensor in Exercise 15.

$$= 2\Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu}.$$

Recalling that $g^{\alpha\gamma}g_{\alpha\nu} = g^{\gamma}{}_{\nu} = \delta^{\gamma}{}_{\nu}$, we divide both sides by 2 and multiply by $g^{\alpha\gamma}$, arriving at Equation 5.75:

$$\begin{aligned}\frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \frac{2}{2}g^{\alpha\gamma}g_{\alpha\nu}\Gamma^{\nu}{}_{\beta\mu} \\ &= \Gamma^{\nu}{}_{\beta\mu}\end{aligned}$$

17 Show how $V^{\beta}{}_{,\alpha}$ and $V^{\mu}\Gamma^{\beta}{}_{\upsilon\alpha}$ transform under change of coordinates. Neither follows a tensor transformation law, but their *sum* does.

$$\begin{aligned}V^{\alpha'}{}_{,\beta'} &= \frac{\partial V^{\alpha'}}{\partial x^{\beta'}} = \Lambda^{\beta}{}_{\beta'}\frac{\partial}{\partial x^{\beta}}\left[\Lambda^{\alpha'}{}_{\alpha}V^{\alpha}\right] \\ &= \Gamma^{\beta}{}_{\beta'}\left[V^{\alpha}\frac{\partial}{\partial x^{\beta}}\Lambda^{\alpha'}{}_{\alpha} + \Lambda^{\alpha'}{}_{\alpha}\frac{\partial}{\partial x^{\beta}}V^{\alpha}\right] \\ &= \Lambda^{\beta}{}_{\beta'}V^{\alpha}\Lambda^{\alpha'}{}_{\alpha,\beta} + \Lambda^{\beta}{}_{\beta'}\Lambda^{\alpha'}{}_{\alpha}V^{\alpha}{}_{,\beta} \\ &\neq \Lambda^{\beta}{}_{\beta'}\Lambda^{\alpha'}{}_{\alpha}V^{\alpha}{}_{,\beta} \\ \frac{\partial\vec{e}_{\alpha'}}{\partial x^{\beta'}} &= \Lambda^{\beta}{}_{\beta'}\frac{\partial}{\partial x^{\beta}}[\Lambda^{\alpha}{}_{\alpha'}\vec{e}_{\alpha}] \\ &= \Lambda^{\beta}{}_{\beta'}\left[\Lambda^{\alpha}{}_{\alpha'}\frac{\partial}{\partial x^{\beta}}\vec{e}_{\alpha} + \vec{e}_{\alpha}\frac{\partial}{\partial x^{\beta}}\Lambda^{\alpha}{}_{\alpha'}\right] \\ &= \Lambda^{\beta}{}_{\beta'}\Lambda^{\alpha}{}_{\alpha'}\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} + \Lambda^{\beta}{}_{\beta'}\Lambda^{\alpha}{}_{\alpha',\beta}\vec{e_{\alpha}} \\ &\neq \Lambda^{\beta}{}_{\beta'}\Lambda^{\alpha}{}_{\alpha'}\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu},\end{aligned}$$

so we have shown that $\partial\vec{e}_{\alpha'}\Big/\partial x^{\beta'}$ is not a tensor, and since V^{μ} is a tensor, and the product of a tensor and a non-tensor is also not a tensor, then $V^{\mu}\Gamma^{\beta}{}_{\nu\alpha}$ is not a tensor.

According to Carroll, the precise transformation is

$$\Gamma^{\nu'}{}_{\mu'\lambda'} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\lambda}{}_{\lambda'}\Lambda^{\nu'}{}_{\nu}\Gamma^{\nu}{}_{\mu\lambda} + \Lambda^{\mu}{}_{\mu'}\Lambda^{\lambda}{}_{\lambda'}\Lambda^{\nu'}{}_{\mu\lambda}.$$

Now we add the two expressions, in order to show that it is a tensor equation

$$\begin{aligned}V^{\nu'}{}_{,\lambda'} + V^{\mu'}\Gamma^{\nu'}{}_{\mu'\lambda'} &= \Lambda^{\lambda}{}_{\lambda'}V^{\nu}\Lambda^{\nu'}{}_{\nu,\lambda} + \Lambda^{\lambda}{}_{\lambda'}\Lambda^{\nu'}{}_{\nu}V^{\nu}{}_{,\lambda} + \Lambda^{\lambda}{}_{\lambda'}\Lambda^{\nu'}{}_{\nu}V^{\mu}\Gamma^{\nu}{}_{\mu\lambda} + \Lambda^{\lambda}{}_{\lambda'}V^{\nu}\Lambda^{\nu'}{}_{\lambda,\mu} \\ &= \Lambda^{\lambda}{}_{\lambda'}\Lambda^{\nu'}{}_{\nu}\left(V^{\nu}{}_{,\lambda} + V^{\mu}\Gamma^{\nu}{}_{\mu\lambda}\right)\end{aligned}$$

So it does in fact transform like a tensor equation, meaning $V^{\nu}{}_{;\lambda}$ is a tensor!

18

Verify Equation 5.78:

$$\left.\begin{aligned}\vec{e}_{\hat{\alpha}}\cdot\vec{e}_{\hat{\beta}} \equiv g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}} \\ \tilde{\omega}^{\hat{\alpha}}\cdot\tilde{\omega}^{\hat{\beta}} \equiv g^{\hat{\alpha}\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}}\end{aligned}\right\}$$

For the basis *vectors*, we have

$$g_{\hat{r}\hat{r}} = \vec{e}_{\hat{r}} \cdot \vec{e}_{\hat{r}} = \vec{e}_r \cdot \vec{e}_r = g_{rr} = 1$$
$$g_{\hat{\theta}\hat{\theta}} = \vec{e}_{\hat{\theta}} \cdot \vec{e}_{\hat{\theta}} = \left(\frac{1}{r}\vec{e}_\theta\right) \cdot \left(\frac{1}{r}\vec{e}_\theta\right) = \frac{1}{r^2}(\vec{e}_\theta \cdot \vec{e}_\theta) = \frac{1}{r}g_{r\theta} = 1$$
$$g_{\hat{r}\hat{\theta}} = \vec{e}_{\hat{r}} \cdot \vec{e}_{\hat{\theta}} = \vec{e}_r \cdot \left(\frac{1}{r}\vec{e}_\theta\right) = \frac{1}{r}(\vec{e}_r \cdot \vec{e}_\theta) = \frac{1}{r}g_{r\theta} = 0$$
$$g_{\hat{\theta}\hat{r}} = g_{\hat{r}\hat{\theta}} = 0$$

So it is indeed true that $g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}}$.

Now for the basis *one-forms*, we have

$$g^{\hat{r}\hat{r}} = \tilde{\omega}^{\hat{r}} \cdot \tilde{\omega}^{\hat{r}} = \tilde{\mathrm{d}}r \cdot \tilde{\mathrm{d}}r = g^{rr} = 1$$
$$g^{\hat{\theta}\hat{\theta}} = \tilde{\omega}^{\hat{\theta}} \cdot \tilde{\omega}^{\hat{\theta}} = (r\tilde{\mathrm{d}}\theta) \cdot (r\tilde{\mathrm{d}}\theta) = r^2(\tilde{\mathrm{d}}\theta \cdot \tilde{\mathrm{d}}\theta) = r^2 g^{\theta\theta} = r^2(1/r^2) = 1$$
$$g^{\hat{r}\hat{\theta}} = \tilde{\omega}^{\hat{r}} \cdot \tilde{\omega}^{\hat{\theta}} = \tilde{\mathrm{d}}r \cdot (r\tilde{\mathrm{d}}\theta) = r(\tilde{\mathrm{d}}r \cdot \tilde{\mathrm{d}}\theta) = r g^{r\theta} = 0$$
$$g^{\hat{\theta}\hat{r}} = g^{\hat{r}\hat{\theta}} = 0$$

So it is indeed true that $g^{\hat{\alpha}\hat{\beta}} = \delta^{\hat{\alpha}\hat{\beta}}$.

19 Repeat the calculations going from Equations 5.81 to 5.84, with $\tilde{\mathrm{d}}r$ and $\tilde{\mathrm{d}}\theta$ as your bases. Show that they form a coordinate basis.

$$\tilde{\mathrm{d}}r = \cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y = \frac{\partial\xi}{\partial x}\tilde{\mathrm{d}}x + \frac{\partial\xi}{\partial y}\tilde{\mathrm{d}}y$$
$$\frac{\partial\xi}{\partial x} = \cos\theta; \quad \frac{\partial\xi}{\partial y} = \sin\theta$$
$$\frac{\partial}{\partial y}\frac{\partial\xi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\xi}{\partial y} \implies \frac{\partial}{\partial y}(x/r) = \frac{\partial}{\partial x}(y/r),$$

which is true, so we have shown that at least $\tilde{\mathrm{d}}r$ may be part of a coordinate basis.

$$\tilde{\mathrm{d}}\theta = -\frac{1}{r}\sin\theta\tilde{\mathrm{d}}x + \frac{1}{r}\cos\theta\tilde{\mathrm{d}}y = \frac{\partial\eta}{\partial x}\tilde{\mathrm{d}}x + \frac{\partial\eta}{\partial y}\tilde{\mathrm{d}}y$$
$$\frac{\partial\eta}{\partial x} = -\frac{1}{r}\sin\theta; \quad \frac{\partial\eta}{\partial y} = \frac{1}{r}\cos\theta$$
$$\frac{\partial}{\partial y}\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\eta}{\partial y} \implies \frac{\partial}{\partial y}\left[-\frac{1}{r}\sin\theta\right] = \frac{\partial}{\partial x}\left[\frac{1}{r}\cos\theta\right],$$

which is also true, and thus we have shown that $\tilde{\mathrm{d}}r$ and $\tilde{\mathrm{d}}\theta$ form a coordinate basis.

20 For a non-coordinate basis $\{\vec{e}_\mu\}$, let $c^\alpha{}_{\mu\nu} = \boldsymbol{\nabla}_{\vec{e}_\mu}\vec{e}_\nu - \boldsymbol{\nabla}_{\vec{e}_\nu}\vec{e}_\mu$. Use this in place of Equation 5.74 to derive a more general expression for Equation 5.75.

$\boldsymbol{c}$ is antisymmetric w.r.t. its bottom indices.

$$c^\alpha{}_{\mu\nu}\vec{e}_\alpha + c^\alpha{}_{\nu\mu}\vec{e}_\alpha = (\boldsymbol{\nabla}_{\vec{e}_\mu}\vec{e}_\nu - \boldsymbol{\nabla}_{\vec{e}_\nu}\vec{e}_\mu) + (\boldsymbol{\nabla}_{\vec{e}_\nu}\vec{e}_\mu - \boldsymbol{\nabla}_{\vec{e}_\mu}\vec{e}_\nu) = 0$$

$$\implies c^{\alpha}{}_{\mu\nu}\vec{e}_{\alpha} = -c^{\alpha}{}_{\nu\mu}\vec{e}_{\alpha}$$
$$\implies c^{\alpha}{}_{\mu\nu} = -c^{\alpha}{}_{\nu\mu}$$

Expanding the covariant derivatives in the original expression, we get

$$\begin{aligned} c^{\alpha}{}_{\mu\nu}\vec{e}_{\alpha} &= \vec{e}_{\nu;\mu} - \vec{e}_{\mu;\nu} \\ &= (\vec{e}_{\nu,\mu} - \vec{e}_{\alpha}\Gamma^{\alpha}{}_{\nu\mu}) - (\vec{e}_{\mu,\nu} - \vec{e}_{\alpha}\Gamma^{\alpha}{}_{\mu\nu}) \\ &= \vec{e}_{\alpha}(\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}) \\ c^{\alpha}{}_{\mu\nu} &= \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu} \end{aligned}$$

Now we recall the result from Exercise 16, but without assuming symmetry of the Christoffel symbols

$$\begin{aligned} g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} &= \Gamma^{\nu}{}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta}g_{\alpha\nu} - \Gamma^{\nu}{}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}{}_{\mu\alpha}g_{\beta\nu} \\ &= \Gamma^{\nu}{}_{\alpha\mu}g_{\beta\nu} + \Gamma^{\nu}{}_{\beta\mu}g_{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}{}_{\mu\beta}g_{\alpha\nu} - \Gamma^{\nu}{}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}{}_{\mu\alpha}g_{\beta\nu} \\ &= g_{\beta\nu}(\Gamma^{\nu}{}_{\alpha\mu} - \Gamma^{\nu}{}_{\mu\alpha}) + g_{\alpha\nu}(\Gamma^{\nu}{}_{\beta\mu} + \Gamma^{\nu}{}_{\mu\beta}) + g_{\nu\mu}(\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\beta\alpha}) \\ &= g_{\beta\nu}c^{\nu}{}_{\alpha\mu} + g_{\alpha\nu}(\Gamma^{\nu}{}_{\beta\mu} + \Gamma^{\nu}{}_{\mu\beta} + \Gamma^{\nu}{}_{\beta\mu} - \Gamma^{\nu}{}_{\beta\mu}) + g_{\nu\mu}c^{\nu}{}_{\alpha\beta} \\ &= g_{\beta\nu}c^{\nu}{}_{\alpha\mu} + g_{\nu\mu}c^{\nu}{}_{\alpha\beta} + g_{\alpha\nu}(2\Gamma^{\nu}{}_{\beta\mu} + c^{\nu}{}_{\mu\beta}) \\ g^{\nu\mu}2g_{\alpha\nu}\Gamma^{\nu}{}_{\beta\mu} &= g^{\nu\mu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c_{\beta\alpha\mu} - c_{\mu\alpha\beta} - c_{\alpha\mu\beta}) \\ \Gamma^{\nu}{}_{\beta\alpha} &= \frac{1}{2}g^{\nu\mu}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} - c_{\beta\alpha\mu} - c_{\mu\alpha\beta} - c_{\alpha\mu\beta}) \end{aligned}$$

21 A uniformly accelerated observer has world line

$$t(\lambda) = a\sinh\lambda, \quad x(\lambda) = a\cosh\lambda$$

(a) Show that the spacelike line tangent to his world line (which is parameterized by λ) is orthogonal to the line parameterized by a.

The line tangent to his world line is

$$\vec{V} \to \frac{\mathrm{d}}{\mathrm{d}\lambda}(t, x) = (a\cosh\lambda, a\sinh\lambda).$$

The line parameterized by a is

$$\vec{W} \to \frac{\mathrm{d}}{\mathrm{d}a}(t, x) = (\sinh\lambda, \cosh\lambda)$$

If they are orthogonal, then their dot product must be zero

$$\vec{V} \cdot \vec{W} = -(a\cosh\lambda\sinh\lambda) + (a\sinh\lambda\cosh\lambda) = 0,$$

which it is.

(b) To prove that this defines a valid coordinate transform from (λ, a) to (t, x), we show that the determinant of the transformation matrix is non-zero.

$$\det \begin{pmatrix} \partial t/\partial\lambda & \partial t/\partial a \\ \partial x/\partial\lambda & \partial x/\partial a \end{pmatrix} = \frac{\partial t}{\partial\lambda}\frac{\partial x}{\partial a} - \frac{\partial t}{\partial a}\frac{\partial x}{\partial\lambda}$$
$$= a\cosh^2\lambda - a\sinh^2\lambda = a$$
$$\neq 0,$$

and so it is indeed a valid coordinate transform.

To plot the curves parameterized by a, we take

$$-t^2 + x^2 = a^2(\cosh^2\lambda - \sinh^2\lambda)$$
$$= a^2,$$

which gives us a family of space-like hyperbola, depending on the chosen value of a.

To plot the curves parameterized by λ, we take

$$x = a\cosh\lambda \implies a = x/\cosh\lambda$$
$$t = a\sinh\lambda = x\sinh\lambda/\cosh\lambda = x\tanh\lambda,$$

which gives us a family of space-like lines, depending on the chosen value of λ.

A plot of these curves is given in Figure 5.2, from which it is clear that only half of the t–x plane is covered. When $|t| = |x|$, then $a = 0$, since $-t^2 + x^2 = a^2$. We already found that the determinant of the coordinate transformation is a, so this would make the determinant 0, making it singular.

(c) Find the metric tensor and Christoffel symbols in (λ, a) coordinates.

First we find the basis vectors:

$$\vec{e}_\lambda = a(\cosh\lambda\vec{e}_t + \sinh\lambda\vec{e}_x),$$
$$\vec{e}_a = \sinh\lambda\vec{e}_t + \cosh\lambda\vec{e}_x.$$

Now we find the components of the metric tensor $\mathbf{g}$ as

$$g_{\lambda\lambda} = a^2(\cosh\lambda\vec{e}_t + \sinh\lambda\vec{e}_x)^2$$
$$= a^2(\cosh^2\lambda\eta_{tt} + \sinh^2\lambda\eta_{xx} + 2\sinh\lambda\cosh\lambda\eta_{tx})$$
$$= a^2(\sinh^2\lambda - \cosh^2\lambda)$$
$$= -a^2$$
$$g_{aa} = (\sinh\lambda\vec{e}_t + \cosh\lambda\vec{e}_x)^2$$
$$= \sinh^2\lambda\eta_{tt} + \cosh^2\lambda\eta_{xx} + 2\sinh\lambda\cosh\lambda\eta_{tx}$$
$$= 1$$

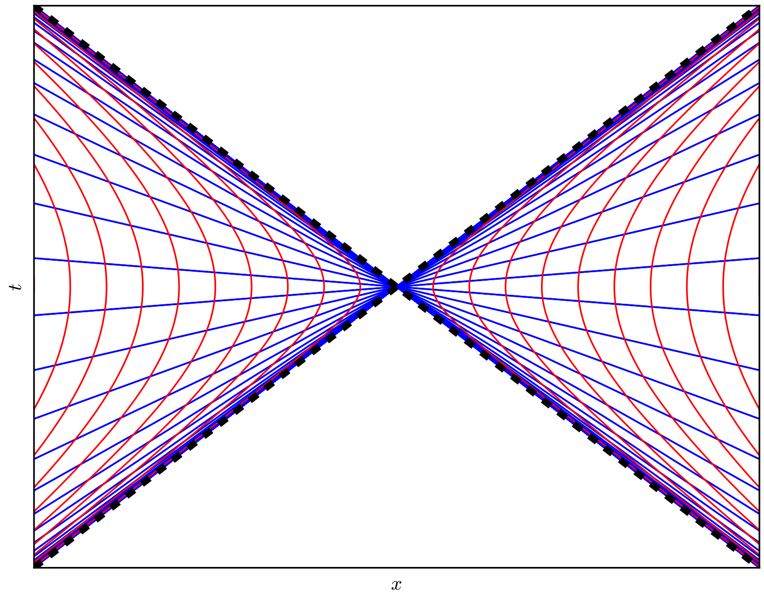


Figure 5.2: Lines of constant λ and a in Problem 21.

$$\begin{aligned} g_{\lambda a} = g_{a\lambda} &= a(\cosh\lambda\vec{e}_t + \sinh\lambda\vec{e}_x)(\sinh\lambda\vec{e}_t + \cosh\lambda\vec{e}_x) \\ &= a(\cosh\lambda\sinh\lambda(\eta_{tt} + \eta_{xx}) + 2\sinh\lambda\cosh\lambda\eta_{tx}) \\ &= 0 \\ \mathbf{g} \underset{(\lambda,a)}{\rightarrow} &\begin{pmatrix} -a^2 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now for the Christoffel symbols, since we know this is a coordinate basis, we can use

$$\Gamma^{\gamma}{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

$$\begin{aligned} \Gamma^{\lambda}{}_{\lambda\lambda} &= \frac{1}{2} g^{\alpha\lambda}(g_{\alpha\lambda,\lambda} + g_{\alpha\lambda,\lambda} - g_{\lambda\lambda,\alpha}) = \frac{1}{2} g^{a\lambda}(-g_{\lambda\lambda,a}) \\ &= 0 \\ \Gamma^{a}{}_{aa} &= \frac{1}{2} g^{\alpha a}(g_{\alpha a,a} + g_{\alpha a,a} - g_{aa,\alpha}) \\ &= 0 \\ \Gamma^{\lambda}{}_{\lambda a} &= \frac{1}{2} g^{\alpha\lambda}(g_{\alpha\lambda,a} + g_{\alpha a,\lambda} - g_{\lambda a,\alpha}) = \frac{1}{2} g^{\lambda\lambda} g_{\lambda\lambda,a} = \frac{1}{2}(-a^{-2})(-2a) \\ &= 1/a \\ \Gamma^{a}{}_{\lambda a} &= \frac{1}{2} g^{\alpha a}(g_{\alpha\lambda,a} + g_{\alpha a,\lambda} - g_{\lambda a,\alpha}) = \frac{1}{2} g^{\lambda a} g_{\lambda\lambda,a} \end{aligned}$$

$$
\begin{aligned}
&= 0 \\
\Gamma^{\lambda}{}_{aa} &= \frac{1}{2} g^{\alpha\lambda}(g_{\alpha a,a} + g_{\alpha a,a} - g_{aa,\alpha}) \\
&= 0 \\
\Gamma^{a}{}_{\lambda\lambda} &= \frac{1}{2} g^{\alpha a}(g_{\alpha\lambda,\lambda} + g_{\alpha\lambda,\lambda} - g_{\lambda\lambda,\alpha}) = \frac{1}{2} g^{aa}(-g_{\lambda\lambda,a}) = \frac{1}{2} \cdot 2 \cdot a \\
&= a
\end{aligned}
$$

22

$$
\begin{aligned}
U^{\alpha}\boldsymbol{\nabla}_{\alpha}V^{\beta} = W^{\beta} &\implies U^{\alpha}V^{\gamma}{}_{;\alpha} = W^{\gamma} \\
&\implies g_{\alpha\beta}U^{\alpha}V^{\gamma}{}_{;\alpha} = g_{\gamma\beta}W^{\gamma} \\
&\implies U^{\alpha}V_{\beta;\alpha} = W_{\beta} \\
&\implies U^{\alpha}\boldsymbol{\nabla}_{\alpha}V_{\beta} = W_{\beta}
\end{aligned}
$$

Chapter 6

Curved Manifolds

6.9 Exercises

1 Determine if the following sets are manifolds, and why. List any exceptional points.

(a) Phase space in Hamiltonian mechanics is generally smooth, though it may contain singular points, depending on the system described. So it is a manifold, excluding the singularities.

(b) The interior of a circle in 2D Euclidean space is smooth everywhere, and is therefore a manifold.

(c) The set of permutations of n objects is not a manifold, as it is discontinuous.

(d) The set of solutions to $xy(x^2 + y^2 - 1)$ is a manifold. The solutions form a unit circle, ($x^2 + y^2 = 1$), as well as lines which span the x- and y-axes ($x = 0$, $y = 0$). The singular values occur at the points of intersection: $(0, 0)$, $(0, \pm 1)$, and $(\pm 1, 0)$.

2 On which of the manifolds in Exercise 1 is it customary to use a metric? What are those metrics? Why would metrics not be defined for some?

(a) Phase space is comprised of two variables, p and q, each of which represent different physical quantities, with incompatible units. For instance, if p is momentum and q is position, then $p^2 + q^2$ is non-physical.

(b) The metric for the interior of a circle in 2D Euclidean space would be the Euclidean norm in 2 dimensions. While this could be given by $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$, it would be more natural to express in units of r and θ.

$$
\begin{aligned}
(\Delta s)^2 &= (\Delta x)^2 + (\Delta y)^2 \\
&= (x - x_0)^2 + (y - y_0)^2 \\
&= r^2\Big[(\cos\theta - \cos\theta_0)^2 + (\sin\theta - \sin\theta_0)^2\Big] \\
&= r^2\Big[\cos^2\theta + \cos^2\theta_0 - 2\cos\theta\cos\theta_0 + \sin^2\theta + \sin^2\theta_0 - 2\sin\theta\sin\theta_0\Big] \\
&= r^2\big[1 + 1 - 2\cos(\theta - \theta_0)\big] = 2r^2\big[1 - \cos(\Delta\theta)\big] \\
&= 4r^2\sin^2(\Delta\theta/2)
\end{aligned}
$$

(c) This was not a manifold.

(d) Since this is again 2D Euclidean space, we could use the Euclidean norm in 2 dimensions. This time it would be more natural to express distances in (x, y) coordinates, unless we restricted ourselves to the unit circle portion of this manifold.

4 Prove the following:

(a) The number of independent terms in $\partial^2 x^\alpha \Big/ \partial x^{\gamma'} \partial x^{\mu'} \, |_{\mathcal{P}}$ is 40.

The total number of components is 4^3, however, we do not want to consider duplicate terms. To find the number of duplicate terms in total, we find the number of duplicate terms for a fixed value of α, and then multiply that by 4. The number of terms for a fixed α is 4^2, and of those, 4 are completely independent (the diagonals), and the remainder exist in pairs. Since we only want one from each pair, we divide the total count by two, which means that the total number of duplicate components is $4\big[(4^2 - 4)/2\big]$, and so the total number of non-duplicate components is $4^3 - 4\big[(4^2 - 4)/2\big] = 40$.

In the next part, I cheat and use a formula. I will apply it to this part first, to show that it works. If you have a symmetric rank k tensor with n dimensions, then it has

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

independent components. In the case of this problem, by fixing α, we get 4 rank 2 tensors of 4 dimensions, and so the total number of independent components is

$$4\binom{4+2-1}{2} = 40.$$

(b) The number for $\partial^2 x^\alpha \Big/ \partial x^{\lambda'} \partial x^{\mu'} \; x^{\nu'} |_{\mathcal{P}}$ is 80.

Here, if we fix α, we have 4 symmetric rank 3 tensors of 4 dimensions, and so there are

$$4\binom{4+3-1}{3} = 80$$

independent components.

(c) The number for $g_{\alpha\beta,\gamma'\mu'}|_{\mathcal{P}}$ is 100.

If we interchange $\alpha\beta$, but fix $\gamma'\mu'$, then we have a symmetric rank 2 tensor of 4 dimensions, which has

$$\binom{4+2-1}{2} = 10$$

independent components. Likewise, if we interchange $\gamma'\mu'$ but fix $\alpha\beta$, we get 10 independent components. Multiply the two and we have 100 independent components.

7

(a) Define $\det(A)$ in terms of cofactors of elements.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} M_{i,j}$$

(b) Compute $\frac{\mathrm{d}}{\mathrm{d}x}\det(A)$, where A is a 2×2 matrix. Show that this satisfies Equation 6.39.

First we note that, for $A_{1\times 1}$, $\det(A) = a_{1,1}$. Thus, for $A_{2\times 2}$, $M_{i,j} = a_{i',j'}$, where $i \neq i'$ and $j \neq j'$. Therefore we can rewrite the determinant of $A_{2\times 2}$ as

$$\begin{aligned}\det(A) &= \sum_{i=1}^{2}(-1)^{i+j}a_{i,j}a_{i',j'} \\ &= (-1)^{j+1}a_{1,j}a_{2,j'} + (-1)^{j+2}a_{2,j}a_{1,j'}.\end{aligned}$$

If we assume $j = 1$ (it doesn't really matter if we choose 1 or 2), then this simplifies to

$$\begin{aligned}\det(A) &= (-1)^2 a_{1,1}a_{2,2} + (-1)^3 a_{2,1}a_{1,2} \\ &= a_{1,1}a_{2,2} - a_{2,1}a_{1,2}.\end{aligned}$$

We can then see that the derivative is

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\det(A) &= \frac{\partial}{\partial x^\mu}(a_{11}a_{22} - a_{21}a_{12}) \\ &= a_{11}a_{22,\mu} + a_{22}a_{11,\mu} - a_{21}a_{12,\mu} - a_{12}a_{21,\mu}\end{aligned}$$

Now to relate this to Equation 6.39, we let A be the metric g. Then the derivative of its determinant is

$$\begin{aligned}g_{,\mu} &= g_{11}g_{22,\mu} + g_{22}g_{11,\mu} - g_{21}g_{12,\mu} - g_{12}g_{21,\mu} \\ &= g_{11}g_{22,\mu} + g_{22}g_{11,\mu} - 2g_{12}g_{12,\mu}.\end{aligned}$$

Now if we expand Equation 6.39, we see we have

$$\begin{aligned}g &= g_{11}g_{22} - g_{12}g_{21} = g_{11}g_{22} - (g_{12})^2 \\ g^{\alpha\beta}g_{\alpha\beta,\mu} &= g^{11}g_{11,\mu} + g^{22}g_{22,\mu} + 2g^{12}g_{12,\mu} \\ gg^{\alpha\beta}g_{\alpha\beta,\mu} &= (g_{11}g_{22} - (g_{12})^2)(g^{11}g_{11,\mu} + g^{22}g_{22,\mu} + 2g^{12}g_{12,\mu}) \\ &= g_{22}g_{11,\mu} - g^{11}(g_{12})^2 g_{11,\mu} + g_{11}g_{22,\mu} - g^{22}(g_{12})^2 g_{22,\mu} + 2g_{11}g_{22}g^{12}g_{12,\mu} - 2g_{12}g_{12,\mu} \\ &= g_{11}g_{22,\mu} + g_{22}g_{11,\mu} - 2g_{12}g_{12,\mu} + 2g_{11}g_{22}g^{12}g_{12,\mu} - (g_{12})^2(g^{11}g_{11,\mu} + g^{22}g_{22,\mu}).\end{aligned}$$

If it is the case that $2g_{11}g_{22}g^{12}g_{12,\mu} - (g_{12})^2(g^{11}g_{11,\mu} + g^{22}g_{22,\mu}) = 0$, then this is consistent with our previous expression for $g_{,\mu}$, but I'm not sure how to show that.

10 A "straight line" on a sphere forms a great circle. The sum of the interior angles of a triangle whose sides are formed by arcs of great circles is greater than 180°. Show that the rotation of a vector, parallel transported around such a triangle (Figure 6.3 in Schutz), is exactly equal to the excess of that 180° sum.

11 What guarantees we can find a vector field $\vec{V}$ satisfying:

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\mu\beta}V^\mu = 0$$

(a) The integrability condition follows from the commuting of partial derivatives, $\left[\partial_\nu, \partial_\beta\right] V^\alpha = 0$. Show that this implies

$$({\Gamma^\alpha}_{\mu\beta,\nu} - {\Gamma^\alpha}_{\mu\nu,\beta}) V^\mu = ({\Gamma^\alpha}_{\mu\beta}{\Gamma^\mu}_{\sigma\nu} - {\Gamma^\alpha}_{\mu\nu}{\Gamma^\mu}_{\sigma\beta}) V^\sigma = 0$$

Since we must satisfy ${V^\alpha}_{,\beta} + {\Gamma^\alpha}_{\mu\beta} V^\mu = 0$, then it must be the case that ${V^\alpha}_{,\beta} = -{\Gamma^\alpha}_{\mu\beta} V^\mu$. Differentiating both sides, we get

$$\begin{aligned}
{V^\alpha}_{,\beta\nu} &= -{\Gamma^\alpha}_{\mu\beta,\nu} V^\mu - {\Gamma^\alpha}_{\mu\beta} {V^\mu}_{,\nu} \\
&= -{\Gamma^\alpha}_{\mu\beta,\nu} V^\mu + {\Gamma^\alpha}_{\mu\beta} {\Gamma^\mu}_{\lambda\nu} V^\lambda \\
{V^\alpha}_{,\nu\beta} &= -{\Gamma^\alpha}_{\mu\nu,\beta} V^\mu + {\Gamma^\alpha}_{\mu\nu} {\Gamma^\mu}_{\sigma\beta} V^\sigma \\
{V^\alpha}_{,\beta\nu} &= {V^\alpha}_{,\nu\beta} \\
\implies -{\Gamma^\alpha}_{\mu\beta,\nu} V^\mu + {\Gamma^\alpha}_{\mu\beta} {\Gamma^\mu}_{\sigma\nu} V^\sigma &= -{\Gamma^\alpha}_{\mu\nu,\beta} V^\mu + {\Gamma^\alpha}_{\mu\nu} {\Gamma^\mu}_{\sigma\beta} V^\sigma \\
({\Gamma^\alpha}_{\mu\beta,\nu} - {\Gamma^\alpha}_{\mu\nu,\beta}) V^\mu &= ({\Gamma^\alpha}_{\mu\beta}{\Gamma^\mu}_{\sigma\nu} - {\Gamma^\alpha}_{\mu\nu}{\Gamma^\mu}_{\sigma\beta}) V^\sigma
\end{aligned}$$

(b) By relabeling indices, we can work this into another form:

$$\begin{aligned}
({\Gamma^\alpha}_{\mu\beta,\nu} - {\Gamma^\alpha}_{\mu\nu,\beta}) V^\mu &= ({\Gamma^\alpha}_{\sigma\beta}{\Gamma^\sigma}_{\mu\nu} - {\Gamma^\alpha}_{\sigma\nu}{\Gamma^\sigma}_{\mu\beta}) V^\mu \\
({\Gamma^\alpha}_{\mu\beta,\nu} - {\Gamma^\alpha}_{\mu\nu,\beta} + {\Gamma^\alpha}_{\sigma\nu}{\Gamma^\sigma}_{\mu\beta} - {\Gamma^\alpha}_{\sigma\beta}{\Gamma^\sigma}_{\mu\nu}) V^\mu &= 0
\end{aligned}$$

13

(a) Show that if $\vec{A}$ and $\vec{B}$ are parallel transported along a curve, their dot product is constant along that curve.

The dot product being constant along the curve means that *it* must be parallel transported along the curve, i.e. $\boldsymbol{\nabla}_{\vec{U}}(\vec{A}\cdot\vec{B}) = 0$. We will now show this.

$$\begin{aligned}
\boldsymbol{\nabla}_{\vec{U}}(\vec{A}\cdot\vec{B}) &= U^\lambda \boldsymbol{\nabla}_\lambda (g_{\alpha\beta} A^\alpha B^\beta) \\
&= U^\lambda (A^\alpha B^\beta \cancel{\boldsymbol{\nabla}_\lambda g_{\alpha\beta}} + g_{\alpha\beta} B^\beta \boldsymbol{\nabla}_\lambda A^\alpha + g_{\alpha\beta} A^\alpha \boldsymbol{\nabla}_\lambda B^\beta) \\
&= B^\beta U^\lambda \boldsymbol{\nabla}_\lambda A^\alpha + A^\alpha U^\lambda \boldsymbol{\nabla}_\lambda B^\beta.
\end{aligned}$$

Notice that the terms $U^\lambda \boldsymbol{\nabla}_\lambda A^\alpha$ and $U^\lambda \boldsymbol{\nabla}_\lambda B^\beta$ are just the parallel transport equations, and so they come out to be zero, meaning $\boldsymbol{\nabla}_{\vec{U}}(\vec{A}\cdot\vec{B}) = 0$, i.e. the dot product is constant along the curve.

(b) Show that if a geodesic is spacelike, timelike, or null *somewhere*, then it remains that way *everywhere*.

Since the dot product of two parallel transported vectors is constant, if we parallel transport a curve's tangent vector along itself (the geodesic), its magnitude $(\vec{U}\cdot\vec{U})$ should remain constant. Since its magnitude doesn't change, it will remain spacelike, timelike, or null.

14 Show that if the curve in Equation 6.8 is a geodesic, the proper length is an affine parameter.

Equation 6.8 states

$$\ell = \int_{\lambda_0}^{\lambda_1} \sqrt{\left|\vec{V} \cdot \vec{V}\right|} \mathrm{d}\lambda.$$

If the curve is a geodesic, we have just shown that the dot product of any two vectors remains constant along the curve, and so we may pull it out of the integral.

$$\ell = \sqrt{\left|\vec{V} \cdot \vec{V}\right|} \int_{\lambda_0}^{\lambda_1} \mathrm{d}\lambda = \sqrt{\left|\vec{V} \cdot \vec{V}\right|} (\lambda_1 - \lambda_0),$$

and so the proper length ℓ is indeed an affine parameter, as it can be obtained by a linear transformation of the parameter of the curve, λ.

16

(a) Derive Equations 6.59 and 6.60 from 6.68.

Somehow Schutz uses a Taylor expansion to get 6.59 from 6.68. I honestly have no idea how he does this, and Taylor expanding vectors and Christoffel symbols is black magic to me, so here's my (obviously wrong) attempt.

$$\begin{aligned}
\delta V^\alpha &= \int_{x^1=a} \Gamma^\alpha{}_{\mu 2} V^\mu \,\mathrm{d}x^2 - \int_{x^1=a+\delta a} \Gamma^\alpha{}_{\mu 2} V^\mu \,\mathrm{d}x^2 \\
&\quad + \int_{x^2=b} \Gamma^\alpha{}_{\mu 1} V^\mu \,\mathrm{d}x^1 - \int_{x^2=b+\delta b} \Gamma^\alpha{}_{\mu 1} V^\mu \,\mathrm{d}x^1 \\
&\approx \int_b^{b+\delta b} \left[\Gamma^\alpha{}_{\mu 2} V^\mu + \delta a \,\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\mu 2} V^\mu\right)\bigg|_a\right] \mathrm{d}x^2 \\
&\quad - \int_b^{b+\delta b} \left[\Gamma^\alpha{}_{\mu 2} V^\mu + \delta a \,\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\mu 2} V^\mu\right)\bigg|_a\right] \mathrm{d}x^2 \\
&\quad + \int_a^{a+\delta a} \left[\Gamma^\alpha{}_{\mu 1} V^\mu + \delta b \,\frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\mu 1} V^\mu\right)\bigg|_b\right] \mathrm{d}x^1 \\
&\quad - \int_a^{a+\delta a} \left[\Gamma^\alpha{}_{\mu 1} V^\mu + \delta b \,\frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\mu 1} V^\mu\right)\bigg|_b\right] \mathrm{d}x^1 \\
&\approx 0
\end{aligned}$$

but then a miracle occurred!!

$$\begin{aligned}
&\approx -\int_b^{b+\delta b} \delta a \,\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\mu 2} V^\mu\right) \mathrm{d}x^2 \\
&\quad + \int_a^{a+\delta a} \delta b \,\frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\mu 1} V^\mu\right) \mathrm{d}x^1
\end{aligned}$$

The next step actually *does* make sense to me. Since we are integrating over such tiny areas (δa and δb), $\int_a^{a+\delta a} f(x) \,\mathrm{d}x \approx \delta a \, f(a)$, so

$$\begin{aligned}
\int_b^{b+\delta b} \delta a \,\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\mu 2} V^\mu\right) \mathrm{d}x^2 &\approx \delta a \,\delta b \,\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\mu 2} V^\mu\right), \\
\int_a^{a+\delta a} \delta b \,\frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\mu 1} V^\mu\right) \mathrm{d}x^1 &\approx \delta a \,\delta b \,\frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\mu 1} V^\mu\right).
\end{aligned}$$

Subtracting the two gives us

$$\delta V^\alpha \approx \delta a\,\delta b\left[-\frac{\partial}{\partial x^1}\left(\Gamma^\alpha{}_{\mu 2}V^\mu\right)+\frac{\partial}{\partial x^2}\left(\Gamma^\alpha{}_{\mu 1}V^\mu\right)\right].$$

(b) Derive Equation 6.61 from this.

Using a generalized form of Equation 6.53:

$$V^\alpha{}_{,\beta} = -\Gamma^\alpha{}_{\mu\beta}V^\mu,$$

we arrive at the expression

$$(\Gamma^\alpha{}_{\nu\lambda}V^\nu)_{,\beta} = \Gamma^\alpha{}_{\nu\lambda,\beta}V^\nu + \Gamma^\alpha{}_{\nu\lambda}V^\nu{}_{,\beta} = \Gamma^\alpha{}_{\nu\lambda,\beta}V^\nu - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\nu{}_{\mu\beta}V^\nu.$$

Now we substitute $\mu \to \nu$ in Equation 6.60, and use this expression to find

$$\begin{aligned}\delta V^\alpha &\approx \delta a\,\delta b\left[-\Gamma^\alpha{}_{\nu 2,1}V^\nu + \Gamma^\alpha{}_{\nu 2}\Gamma^\nu{}_{\mu 1}V^\nu + \Gamma^\alpha{}_{\nu 1,2}V^\nu - \Gamma^\alpha{}_{\nu 1}\Gamma^\nu{}_{\mu 2}V^\nu\right]\\ &\approx \delta a\,\delta b\left[\Gamma^\alpha{}_{\nu 1,2} - \Gamma^\alpha{}_{\nu 2,1} + \Gamma^\alpha{}_{\nu 2}\Gamma^\nu{}_{\mu 1} - \Gamma^\alpha{}_{\nu 1}\Gamma^\nu{}_{\mu 2}\right]V^\nu.\end{aligned}$$

18

(a) Derive Equations 6.69 and 6.70 from 6.68.

$$\begin{aligned}R_{\alpha\beta\mu\nu} &= \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})\\ R_{\beta\alpha\mu\nu} &= \frac{1}{2}(g_{\beta\nu,\alpha\mu} - g_{\beta\mu,\alpha\nu} + g_{\alpha\mu,\beta\nu} - g_{\alpha\nu,\beta\mu})\\ &= \frac{1}{2}(-g_{\alpha\nu,\beta\mu} + g_{\alpha\mu,\beta\nu} - g_{\beta\mu,\alpha\nu} + g_{\beta\nu,\alpha\mu})\\ &= -R_{\alpha\beta\mu\nu}\\ R_{\alpha\beta\nu\mu} &= \frac{1}{2}(g_{\alpha\mu,\beta\nu} - g_{\alpha\nu,\beta\mu} + g_{\beta\nu,\alpha\mu} - g_{\beta\mu,\alpha\nu})\\ &= \frac{1}{2}(-g_{\alpha\nu,\beta\mu} + g_{\alpha\mu,\beta\nu} - g_{\beta\mu,\alpha\nu} + g_{\beta\nu,\alpha\mu})\\ &= -R_{\alpha\beta\mu\nu}\\ R_{\mu\nu\alpha\beta} &= \frac{1}{2}(g_{\mu\beta,\nu\alpha} - g_{\mu\alpha,\nu\beta} + g_{\nu\alpha,\mu\beta} - g_{\nu\beta,\mu\alpha})\\ &= \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})\\ &= R_{\alpha\beta\mu\nu}\end{aligned}$$

$$\begin{aligned}2(R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta}) &= g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}\\ &\quad + g_{\alpha\mu,\nu\beta} - g_{\alpha\beta,\nu\mu} + g_{\nu\beta,\alpha\mu} - g_{\nu\mu,\alpha\beta}\\ &\quad + g_{\alpha\beta,\mu\nu} - g_{\alpha\nu,\mu\beta} + g_{\mu\nu,\alpha\beta} - g_{\mu\beta,\alpha\nu}\end{aligned}$$

$$= 0$$

(b) Show that Equation 6.69 reduces the number of independent components from $4 \times 4 \times 4 \times 4$ to $6 \times 7/2$.

For a rank-2 symmetric tensor, you have $(n/2)(n+1)$ independent components. For an anti-symmetric tensor you have $(n/2)(n-1)$ independent components. So for each of our pairs of anti-symmetric indices, there are $(n/2)(n-1)$ independent components. We can then treat the two pairs as a single pair of symmetric indices, with that many possible values. The number of indices is therefore:

$$(1/2)[(n/2)(n-1)][(n/2)(n-1)+1] = (1/2)[(4/2)(4-1)][(4/2)(4-1)+1] = 6 \times 7/2 = 21.$$

(c) Show that Equation 6.70 only imposes one additional relation, separate from Equation 6.69, reducing the total independent components to 20.

The addition of Equation 6.70 adds the condition that $R_{\alpha[\beta\mu\nu]} = 0$, and so the number of independent components becomes

$$\left(\!\!\binom{4}{3}\!\!\right) = \binom{4+3-1}{3} = \binom{6}{3} = 20.$$

19 Prove that the components of the Riemann tensor are all zero for polar coordinates in the Euclidean plane. Recall that:

$$\Gamma^{\theta}{}_{(\theta r)} = 1/r; \quad \Gamma^{r}{}_{\theta\theta} = -r$$
$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\beta\mu}.$$

According to the computer algebra system, `Maxima`, the components are all zero.

```
(%i1) load(ctensor)$


(%i2) ct_coordsys(polar)$


(%i3) cmetric()$


(%i4) lg;
                                      [ 1  0  ]
(%o4)                                 [       ]
                                      [     2 ]
                                      [ 0  r  ]
(%i5) riemann(mcs);
This spacetime is flat
(%o5)                                   done
```

24 Using Equation 6.88, derive Equation 6.89.

$$\begin{aligned}
R_{\alpha\beta\mu\nu,\lambda} &= \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}) \\
R_{\alpha\beta\lambda\mu,\nu} &= \frac{1}{2}(g_{\alpha\mu,\beta\lambda\nu} - g_{\alpha\lambda,\beta\mu\nu} + g_{\beta\lambda,\alpha\mu\nu} - g_{\beta\mu,\alpha\lambda\nu}) \\
R_{\alpha\beta\nu\lambda,\mu} &= \frac{1}{2}(g_{\alpha\lambda,\beta\nu\mu} - g_{\alpha\nu,\beta\lambda\mu} + g_{\beta\nu,\alpha\lambda\mu} - g_{\beta\lambda,\alpha\nu\mu}) \\
2(R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu}) &= g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda} \\
&\quad + g_{\alpha\mu,\beta\lambda\nu} - g_{\alpha\lambda,\beta\mu\nu} + g_{\beta\lambda,\alpha\mu\nu} - g_{\beta\mu,\alpha\lambda\nu} \\
&\quad + g_{\alpha\lambda,\beta\nu\mu} - g_{\alpha\nu,\beta\lambda\mu} + g_{\beta\nu,\alpha\lambda\mu} - g_{\beta\lambda,\alpha\nu\mu} \\
&= 0
\end{aligned}$$

25

(a) Prove the Ricci tensor is the only independent contraction of the Riemann tensor. All others are $\pm R^{\alpha}{}_{\beta\mu\nu}$ or 0.

There are three possible ways to contract the Riemann tensor. If we contract on the second lower index, we have the definition of the Ricci tensor: $R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu}$.

The value of contracting the last index is the easiest to find, and can be found by manipulating the above expression and invoking the anti-symmetry properties of the Riemann tensor:

$$R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu} = -R^{\alpha}{}_{\beta\nu\alpha} \implies R^{\alpha}{}_{\beta\nu\alpha} = -R_{\beta\nu}.$$

Given this identity, finding the value of the remaining contraction is easy. Equation 6.70 states that

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0.$$

If we raise the α's with the metric, we get

$$\begin{aligned}
g^{\alpha\beta}(R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta}) &= 0 \\
R^{\beta}{}_{\beta\mu\nu} + R^{\beta}{}_{\nu\beta\mu} + R^{\beta}{}_{\mu\nu\beta} &= 0 \\
R^{\beta}{}_{\beta\mu\nu} + R^{\beta}{}_{\nu\beta\mu} - R^{\beta}{}_{\mu\beta\nu} &= 0 \\
R^{\beta}{}_{\beta\mu\nu} &= 0
\end{aligned}$$

(b) Show that the Ricci tensor is symmetric.

$$\begin{aligned}
R_{\beta\nu} &= R^{\alpha}{}_{\beta\alpha\nu} \\
g_{\alpha\lambda}R_{\beta\nu} &= R_{\lambda\beta\alpha\nu} = R_{\alpha\nu\lambda\beta}
\end{aligned}$$

$$g^{\alpha\lambda} g_{\alpha\lambda} R_{\beta\nu} = g^{\alpha\lambda} R_{\alpha\nu\lambda\beta} = R^{\lambda}{}_{\nu\lambda\beta} = R_{\nu\beta}$$
$$\implies R_{\beta\nu} = R_{\nu\beta}$$

28

(a) Derive Equation 6.19 using the coordinate transformation $(x, y, z) \to (r, \theta, \phi)$

We begin by finding the basis vectors in (r, θ, ϕ), using

$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z,$$
$$\vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y + \frac{\partial z}{\partial \theta}\vec{e}_z,$$
$$\vec{e}_\phi = \frac{\partial x}{\partial \phi}\vec{e}_x + \frac{\partial y}{\partial \phi}\vec{e}_y + \frac{\partial z}{\partial \phi}\vec{e}_z.$$

The variables transform according to

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta.$$

Now we take the derivatives

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,$$
$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = -r\sin\theta\cos\phi,$$
$$\frac{\partial z}{\partial r} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta, \quad \frac{\partial z}{\partial \phi} = 0.$$

The basis vectors are therefore

$$\vec{e}_r = \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z$$
$$\vec{e}_\theta = r\cos\theta\cos\phi\vec{e}_x + r\cos\theta\sin\phi\vec{e}_y - r\sin\theta\vec{e}_z$$
$$\vec{e}_\phi = -r\sin\theta\sin\phi\vec{e}_x + r\sin\theta\cos\phi\vec{e}_y$$

Now we find the components of the metric tensor using the fact that $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$.

$$g_{rr} = \vec{e}_r \cdot \vec{e}_r = (\sin\theta\cos\phi)^2\delta_{xx} + (\sin\theta\sin\phi)^2\delta_{yy} + (\cos^2\theta)^2\delta_{zz} + \dots$$
$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = \sin^2\theta + \cos^2\theta$$
$$= 1,$$
$$g_{\theta\theta} = \vec{e}_\theta \cdot \vec{e}_\theta = (r\cos\theta\cos\phi)^2\delta_{xx} + (r\cos\theta\sin\phi)^2\delta_{yy} + (-r\sin\theta)^2\delta_{zz}$$
$$= r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)$$

$$
\begin{aligned}
&= r^2(\cos^2\theta + \sin^2\theta) \\
&= r^2, \\
g_{\phi\phi} = \vec{e}_\phi \cdot \vec{e}_\phi &= (-r\sin\theta\sin\phi)^2\delta_{xx} + (r\sin\theta\cos\phi)^2\delta_{yy} \\
&= r^2(\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi) \\
&= r^2\sin^2\theta.
\end{aligned}
$$

Now for the off-diagonal elements, we take advantage of the symmetry properties of the metric to reduce it from 6 terms to 3.

$$
\begin{aligned}
g_{r\theta} = g_{\theta r} = \vec{e}_r \cdot \vec{e}_\theta &= (\sin\theta\cos\phi)(r\cos\theta\cos\phi)\delta_{xx} + (\sin\theta\sin\phi)(r\cos\theta\sin\phi)\delta_{yy} + (\cos\theta)(-r\sin\theta)\delta_{zz} \\
&= r\Big(\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta\Big) \\
&= 0, \\
g_{r\phi} = g_{\phi r} = \vec{e}_r \cdot \vec{e}_\phi &= (\sin\theta\cos\phi)(-r\sin\theta\sin\phi)\delta_{xx} + (\sin\theta\sin\phi)(r\sin\theta\cos\phi)\delta_{yy} + (\cos\theta)(0)\delta_{zz} \\
&= r(-\sin^2\theta\sin\phi\cos\phi + \sin^2\theta\sin\phi\cos\phi) \\
&= 0, \\
g_{\theta\phi} = g_{\phi\theta} = \vec{e}_\theta \cdot \vec{e}_\phi &= (r\cos\theta\cos\phi)(-r\sin\theta\sin\phi)\delta_{xx} + (r\cos\theta\sin\phi)(r\sin\theta\cos\phi)\delta_{yy} \\
&= r^2(-\cos\theta\cos\phi\sin\theta\sin\phi + \cos\theta\cos\phi\sin\theta\sin\phi) \\
&= 0.
\end{aligned}
$$

The metric tensor in spherical polar coordinates is therefore

$$
(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.
$$

(b) Use Equation 6.19 to find the metric on the surface of a sphere.

On the surface of a sphere, r is fixed, and therefore $\Delta r = 0$. As a result of this, we do not need to consider g_{rr}, and the only relevant components become (θ, ϕ). So we can simplify the metric as:

$$
(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}.
$$

(c) Find the components of $g^{\alpha\beta}$ on the surface of a sphere.

Since $g_{\alpha\beta}$ is a diagonal matrix, the components of its inverse are simply equal to their multiplicative inverse. So the matrix is

$$(g_{ij}) = \begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/r^2\sin^2\theta \end{pmatrix}.$$

29 Calculate the Riemann tensor of the unit sphere in spherical polar coordinates.

The metric for a unit sphere in spherical polars is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

and so one component of the Riemann tensor is

$$\begin{aligned} R_{\theta\phi\theta\phi} &= \frac{1}{2}\big(g_{\theta\phi,\phi\theta} - g_{\theta\theta,\phi\phi} + g_{\phi\theta,\theta\phi} - g_{\phi\phi,\theta\theta}\big) = \frac{1}{2}\big(g_{\theta\theta,\phi\phi} - g_{\phi\phi,\theta\theta}\big) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial\phi^2}1 - \frac{\partial^2}{\partial\theta^2}\sin^2\theta\right) = \frac{1}{2}\sin^2\theta. \end{aligned}$$

Using the symmetry and anti-symmetry properties of the Riemann tensor, we find the remaining components:

$$\begin{aligned} R_{\phi\theta\phi\theta} &= \sin^2\theta \\ R_{\theta\phi\phi\theta} &= R_{\phi\theta\theta\phi} = -\sin^2\theta. \end{aligned}$$

All remaining components are zero, as they have indices $\theta\theta\theta\phi$ or $\phi\phi\phi\theta$, and the only non-zero second derivative of the metric is $g_{\phi\phi,\theta\theta}$, which requires two of each index, not three.

30 Calculate the Riemann tensor on a cylinder.

The metric in cylindrical polars, (r, θ, z), is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the surface of a cylinder (excluding the top and bottom) the radius is unchanging, so $\Delta r =$, as was the case on the surface of a sphere. The metric can therefore be simplified in (θ, z) coordinates as:

$$(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the metric alone, it is obvious that the components of the Riemann tensor must *all* be zero. This is because the Riemann tensor depends on second derivatives of the components of the metric, and the only variable term is $g_{\theta\theta} = r^2$. Since we removed the dependence on the *coordinate* r, *none* of the terms in the Riemann tensor will involve differentiating with respect to r, and therefore they will *all* be zero.

32 A 4D manifold has coodinates (u, v, w, p), and a metric

$$(g_{\alpha\beta}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that the manifold is flat and has signature +2.

Since every element in the metric is a constant, $g_{\alpha\beta,\mu\nu} \equiv 0$, and therefore $R_{\alpha\beta\mu\nu} \equiv 0$, so the manifold is flat.

The signature is just the sum of the diagonal elements, which in this case is $1 + 1 = 2$.

(b) Since this manifold is flat and has signature +2, it must be a Minkowski spacetime. Find a coordinate transformation to (t, x, y, z).

$$\begin{aligned} \Lambda g &= \eta \\ \Lambda g g^{-1} &= \eta g^{-1} \\ \Lambda &= \eta g^{-1} = \eta g \text{ (since } g \text{ is symmetric)} \\ (\Lambda_{\alpha\beta}) &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

33 A three-sphere (or glome) is the 4D analog of a sphere, with cartesian coordinates (x, y, z, w), described by the equation $x^2 + y^2 + z^2 + w^2 = r^2$, where r is its radius.

(a) Define coordinates (r, θ, ϕ, χ), given by

$$x = r \sin\chi \sin\theta \cos\phi, \qquad y = r \sin\chi \sin\theta \sin\phi,$$

$$z = r \sin\chi \cos\theta, \qquad w = r \cos\chi,$$

and show that (θ, ϕ, χ) form the coordinates of the surface of the sphere.

Per usual, we begin by finding the elements of the Jacobian

$$\Lambda : (x, y, z, w) \to (r, \theta, \phi, \chi).$$

$$\begin{aligned}
\partial x/\partial r &= \sin\chi\sin\theta\cos\phi, & \partial y/\partial r &= \sin\chi\sin\theta\sin\phi, & \partial z/\partial r &= \sin\chi\cos\theta, & \partial w/\partial r &= \cos\chi,\\
\partial x/\partial \theta &= r\sin\chi\cos\theta\cos\phi, & \partial y/\partial \theta &= r\sin\chi\cos\theta\sin\phi, & \partial z/\partial \theta &= -r\sin\chi\sin\theta, & \partial w/\partial \theta &= 0,\\
\partial x/\partial \phi &= -r\sin\chi\sin\theta\sin\phi, & \partial y/\partial \phi &= r\sin\chi\sin\theta\cos\phi, & \partial z/\partial \phi &= 0, & \partial w/\partial \phi &= 0,\\
\partial x/\partial \chi &= r\cos\chi\sin\theta\cos\phi, & \partial y/\partial \chi &= r\cos\chi\sin\theta\sin\phi, & \partial z/\partial \chi &= r\cos\chi\cos\theta, & \partial w/\partial \chi &= -r\sin\chi.
\end{aligned}$$

the basis vectors are then

$$\vec{e}_\xi = \frac{\partial x^\alpha}{\partial \xi}\vec{e}_\alpha$$

$$\begin{aligned}
\vec{e}_r &= \sin\chi\sin\theta\cos\phi\vec{e}_x & \vec{e}_\theta &= r\sin\chi\cos\theta\cos\phi\vec{e}_x & \vec{e}_\phi &= -r\sin\chi\sin\theta\sin\phi\vec{e}_x & \vec{e}_\chi &= r\cos\chi\sin\theta\cos\phi\vec{e}_x\\
&+\sin\chi\sin\theta\sin\phi\vec{e}_y & &+r\sin\chi\cos\theta\sin\phi\vec{e}_y & &+r\sin\chi\sin\theta\cos\phi\vec{e}_y & &+r\cos\chi\sin\theta\sin\phi\vec{e}_y\\
&+\sin\chi\cos\theta\vec{e}_z & &-r\sin\chi\sin\theta\vec{e}_z & & & &+r\cos\chi\cos\theta\\
&+\cos\chi\vec{e}_w & & & & & &-r\sin\chi\vec{e}_w
\end{aligned}$$

Notice that if we fix $\chi = \pi/2$, this reduces to the basis vectors for 2D spherical polars.

The components of the metric can be found using $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$.

$$\begin{aligned}
g_{rr} &= \sin^2\chi\sin^2\theta\cos^2\phi\eta_{xx} + \sin^2\chi\sin^2\theta\sin^2\phi\eta_{yy} + \sin^2\chi\cos^2\theta\eta_{zz} + \cos^2\chi\eta_{ww}\\
&= \sin^2\chi(\sin^2\theta + \cos^2\theta) + \cos^2\chi = \sin^2\chi + \cos^2\chi = 1\\
g_{\theta\theta} &= r^2\Big(\sin^2\chi\cos^2\chi\cos^2\phi\eta_{xx} + \sin^2\chi\cos^2\theta\sin^2\phi\eta_{yy} + \sin^2\chi\sin^2\theta\eta_{zz}\Big)\\
&= r^2\sin^2\chi(\cos^2\theta + \sin^2\theta) = r^2\sin^2\chi\\
g_{\phi\phi} &= r^2\Big(\sin^2\chi\sin^2\theta\sin^2\phi\eta_{xx} + \sin^2\chi\sin^2\theta\cos^2\phi\eta_{yy}\Big)\\
&= r^2\sin^2\chi\sin^2\theta\\
g_{\chi\chi} &= r^2\Big(\cos^2\chi\sin^2\theta\cos^2\phi\eta_{xx} + \cos^2\chi\sin^2\theta\sin^2\phi\eta_{yy} + \cos^2\chi\cos^2\theta\eta_{zz} + \sin^2\chi\eta_{ww}\Big)\\
&= r^2\Big(\cos^2\chi\sin^2\theta + \cos^2\chi\cos^2\theta + \sin^2\chi\Big) = r^2\Big(\cos^2\chi + \sin^2\chi\Big) = r^2
\end{aligned}$$

To show that the off-diagonal terms are zero, I got lazy and used the Maxima computer algebra system. Its naming convention and ordering for these coordinates is different, but it still makes it clear that the metric is diagonal.

```
(%i1) load(ctensor)$            /* load the component tensor package */
(%i2) ct_coordsys(spherical4d)$ /* use the 3-sphere metric */
(%i3) lg;                       /* display the metric */
                [ 1  0         0                    0            ]
                [                                                ]
                [     2                                          ]
```

```
                [ 0  r         0                    0             ]
(%o3)           [                                                 ]
                [            2    2                               ]
                [ 0  0   r  sin (theta)             0             ]
                [                                                 ]
                [                               2        2    2   ]
                [ 0  0         0          sin (eta) r  sin (theta) ]
```

So in our notation, the metric tensor is

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 \sin^2\chi & 0 & 0 \\ 0 & 0 & r^2 \sin^2\chi \sin^2\theta & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}.$$

(b) Show that the metric on the *surface* of the three-sphere only has non-zero components $g_{\theta\theta}$, $g_{\phi\phi}$, and $g_{\chi\chi}$.

On the surface of a three-sphere, r is unchanging, so Δr is always zero. Thus, we may reduce the dimensionality of the metric to 3: (θ, ϕ, χ).

$$(g_{ij}) = \begin{pmatrix} r^2 \sin^2\chi & 0 & 0 \\ 0 & r^2 \sin^2\chi \sin^2\theta & 0 \\ 0 & 0 & r^2 \end{pmatrix}.$$

34 Prove the following identities for a general metric tensor in a general coordinate system. Equations 6.39 and 6.40 will be helpful.

(a) $\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}(\ln|g|)_{,\nu}$

$$\Gamma^\mu{}_{\mu\nu} = \frac{(\sqrt{-g})_{,\nu}}{\sqrt{-g}} = \frac{1}{2\sqrt{-g}}\frac{(-g)_{,\nu}}{\sqrt{-g}} = \frac{(-g)_{,\nu}}{2(-g)} = \frac{|g|_{,\nu}}{2|g|} = \frac{1}{2}(\ln|g|)_{,\nu}$$

(b) $g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} = (-g^{\alpha\beta}\sqrt{-g})_{,\beta}/\sqrt{-g}$

$$\begin{aligned} g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} &= -(g^{\alpha\beta}\sqrt{-g})_{,\beta}/\sqrt{-g} \\ &= -(g^{\alpha\beta}(\sqrt{-g})_{,\beta} + g^{\alpha\beta}{}_{,\beta}\sqrt{-g})/\sqrt{-g} \\ &= -(g^{\alpha\beta}(\sqrt{-g})_{,\beta}/\sqrt{-g} + g^{\alpha\beta}{}_{,\beta}) \\ &= -(g^{\alpha\beta}\Gamma^\lambda{}_{\lambda\beta} + g^{\alpha\beta}{}_{,\beta}) \\ \frac{1}{2}g^{\mu\nu}g^{\beta\alpha}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) &= -(g^{\alpha\beta}g^{\lambda\sigma}g_{\lambda\sigma,\beta}/2 + g^{\alpha\beta}{}_{,\beta}) \\ \frac{1}{2}g^{\mu\nu}g^{\beta\alpha}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu}) - g^{\mu\nu}g^{\beta\alpha}g_{\mu\nu,\beta}/2 &= -(g^{\alpha\beta}g^{\lambda\sigma}g_{\lambda\sigma,\beta}/2 + g^{\alpha\beta}{}_{,\beta}) \end{aligned}$$

$$\frac{1}{2}g^{\mu\nu}g^{\beta\alpha}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu}) = -g^{\alpha\beta}{}_{,\beta}$$
$$-\frac{1}{2}g^{\mu\nu}(g^{\beta\alpha}{}_{,\nu}g_{\beta\mu} + g^{\beta\alpha}{}_{,\mu}g_{\beta\nu}) = -g^{\alpha\beta}{}_{,\beta}$$
$$-\frac{1}{2}(\delta_\beta{}^\nu g^{\beta\alpha}{}_{,\nu} + \delta_\beta{}^\mu g^{\beta\alpha}{}_{,\mu}) = -g^{\alpha\beta}{}_{,\beta}$$
$$-\frac{1}{2}(2g^{\beta\alpha}{}_{,\beta}) = -g^{\alpha\beta}{}_{,\beta}$$

(c) $F^{[\mu\nu]}{}_{;\nu} = (\sqrt{-g}F^{[\mu\nu]})_{,\nu}/\sqrt{-g}$

$$F^{[\mu\nu]}{}_{;\nu} = F^{[\mu\nu]}{}_{,\nu} + F^{[\mu\nu]}\Gamma^\alpha{}_{\nu\alpha} = (F^{[\mu\nu]}{}_{,\nu}\sqrt{-g} + F^{\mu\nu}(\sqrt{-g})_{,\nu})/\sqrt{-g} = (\sqrt{-g}F^{\mu\nu})_{,\nu}$$

(d) $g^{\alpha\sigma}g_{\sigma\beta,\gamma} = -g^{\alpha\sigma}{}_{,\gamma}g_{\sigma\beta}$ We start with $g^{\alpha\sigma}g_{\sigma\beta} = \delta^\alpha{}_\beta$. Then we differentiate both sides to get

$$g^{\alpha\sigma}{}_{,\gamma}g_{\sigma\beta} + g^{\alpha\sigma}g_{\sigma\beta,\gamma} = 0$$
$$g^{\alpha\sigma}g_{\sigma\beta,\gamma} = -g^{\alpha\sigma}{}_{,\gamma}g_{\sigma\beta}$$

(e) $g^{\mu\nu}{}_{,\alpha} = -\Gamma^\mu{}_{\beta\alpha}g^{\beta\nu} - \Gamma^\nu{}_{\beta\alpha}g^{\mu\beta}$

$$g^{\mu\nu}{}_{;\alpha} = g^{\mu\nu}{}_{,\alpha} + \Gamma^\mu{}_{\beta\alpha}g^{\beta\nu} + \Gamma^\nu{}_{\beta\alpha}g^{\mu\beta} = 0$$
$$g^{\mu\nu}{}_{,\alpha} = -\Gamma^\mu{}_{\beta\alpha}g^{\beta\nu} - \Gamma^\nu{}_{\beta\alpha}g^{\mu\beta}$$

35 Compute the metric tensor, Christoffel symbols, and Riemann tensor for a spacetime with line element:

$$\mathrm{d}s^2 = -e^{2\Phi}\,\mathrm{d}t^2 + e^{2\Lambda}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

Based on the line element, the metric must be

$$(g_{\alpha\beta}) = \begin{pmatrix} -e^{2\Phi} & 0 & 0 & 0 \\ 0 & e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad (g^{\alpha\beta}) = \begin{pmatrix} -e^{-2\Phi} & 0 & 0 & 0 \\ 0 & e^{-2\Lambda} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

For the rest of this problem, I took advantage of the `Maxima` computer algebra system. According to it, the non-zero, unique Christoffel symbols are

$$\Gamma^r{}_{tt} = \exp(2\Phi - 2\Lambda)\frac{\mathrm{d}\Phi}{\mathrm{d}r} \qquad \Gamma^t{}_{rt} = \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$
$$\Gamma^r{}_{rr} = \frac{\mathrm{d}\Lambda}{\mathrm{d}r} \qquad \Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r}$$
$$\Gamma^r{}_{\theta\theta} = -\exp(-2\Lambda)r \qquad \Gamma^\phi{}_{\theta\phi} = \cot\theta$$
$$\Gamma^r{}_{\phi\phi} = -\exp(-2\Lambda)r\sin^2\theta \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

The independent non-zero components of the Riemann tensor are

$$R_{t\theta t\theta} = \exp\bigl(2(\Phi-\Lambda)\bigr)\left[\frac{\mathrm{d}\Phi}{\mathrm{d}r}\left(\frac{\mathrm{d}\Lambda}{\mathrm{d}r}-\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)-\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2}\right] \qquad R_{t\theta t\theta} = R_{t\phi t\phi} = -\frac{1}{r}\exp\bigl(2(\Phi-\Lambda)\bigr)\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

$$R_{rrtt} = \frac{\mathrm{d}\Lambda}{\mathrm{d}r}\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} - \left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^2 \qquad R_{r\theta r\theta} = R_{r\phi r\phi} = -\frac{1}{r}\frac{\mathrm{d}\Lambda}{\mathrm{d}r}$$

$$R_{\theta\phi\theta\phi} = \exp(-2\Lambda) - 1 \qquad R_{\phi\phi\theta\theta} = \exp(-2\Lambda)\bigl(\exp(2\Lambda)-1\bigr)\sin^2\theta$$

$$R_{\theta\theta tt} = -r\exp(-2\Lambda)\frac{\mathrm{d}\Phi}{\mathrm{d}r} \qquad R_{\theta\theta rr} = r\exp(-2\Lambda)\frac{\mathrm{d}L}{\mathrm{d}r}$$

36 Consider a 4D manifold with coordinates (t, x, y, z) and line element

$$\mathrm{d}s^2 = -(1+2\phi)\,\mathrm{d}t^2 + (1-2\phi)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2),$$

with $|\phi(t,x,y,z)| \ll 1$. At an arbitrary point $\mathcal{P}$ with coordinates (t_0, x_0, y_0, z_0), find a coordinate transformation to LIF. How does this frame accelerate with respect to the original coordinates? Do all of this to first order in ϕ.

By inspection of the line element, we can see that the metric has components

$$(g_{\alpha\beta}) \underset{(t,x,y,z)}{\rightarrow} \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & (1-2\phi) & 0 & 0 \\ 0 & 0 & (1-2\phi) & 0 \\ 0 & 0 & 0 & (1-2\phi) \end{pmatrix}.$$

We want a transformation to a Minkowski spacetime, i.e.

$$\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}g_{\alpha'\beta'} = \eta_{\alpha\beta}.$$

Now, there may be multiple transformations which satisfy this, so we need only find one. Since both $\boldsymbol{g}$ and $\boldsymbol{\eta}$ are diagonal, I assume that $\boldsymbol{\Lambda}$ is diagonal as well, and find its components.

$$\eta_{00} = \Lambda^{0'}{}_{0}\Lambda^{0'}{}_{0}g_{0'0'} \qquad \eta_{ii} = \Lambda^{i'}{}_{i}\Lambda^{i'}{}_{i}g_{i'i'}$$

$$-1 = (\Lambda^{0'}{}_{0})^2(-(1+2\phi)) \qquad 1 = (\Lambda^{i'}{}_{i})^2(1-2\phi)$$

$$\Lambda^{0'}{}_{0} = (1+2\phi)^{-1/2} \qquad \Lambda^{i'}{}_{i} = (1-2\phi)^{-1/2}$$

Since we know that ϕ is small, we can use the approximation $(1+x)^{-1/2} = (1-x/2) + \mathcal{O}\bigl(x^2\bigr)$, to find

$$\Lambda^{0'}{}_{0} \approx (1-\phi) \qquad \Lambda^{i'}{}_{i} \approx (1+\phi)$$

(39)

Chapter 7

Physics in a curved spacetime

7.6 Exercises

1 If Equation 7.3 were the correct generalization of 7.1 in a curved spacetime, what are the implications? What would happen to the number of particles in a comoving volume of the fluid over time? May we experimentally distinguish between Equations 7.2 and 7.3?

The number of particles would change proportionally to the square of the Ricci scalar, which corresponds to the curvature of the manifold. Whether particles are created (+) or destroyed (−) would depend on the sign of q in the equation.

We could set up some experiment which tests for a change in the number of particles in a moving fluid, in various gravitational fields, to verify whether the RHS of the equation is non-zero.

2 Compute $g^{\alpha\beta}$ for the line element given by Equation 7.8, to first order in ϕ.

Based on the line element, we can infer that the metric is

$$(g_{\alpha\beta}) \underset{(t,x,y,z)}{\to} \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & (1-2\phi) & 0 & 0 \\ 0 & 0 & (1-2\phi) & 0 \\ 0 & 0 & 0 & (1-2\phi) \end{pmatrix}$$

$$(g^{\alpha\beta}) \underset{(t,x,y,z)}{\to} \begin{pmatrix} -(1+2\phi)^{-1} & 0 & 0 & 0 \\ 0 & (1-2\phi)^{-1} & 0 & 0 \\ 0 & 0 & (1-2\phi)^{-1} & 0 \\ 0 & 0 & 0 & (1-2\phi)^{-1} \end{pmatrix} \approx \begin{pmatrix} -(1-2\phi) & 0 & 0 & 0 \\ 0 & (1+2\phi) & 0 & 0 \\ 0 & 0 & (1+2\phi) & 0 \\ 0 & 0 & 0 & (1+2\phi) \end{pmatrix}$$

3 Calculate the Christoffel sybmols for the metric given by Equation 7.8, to first order in ϕ, assuming $\phi = \phi(t,x,y,z)$.

I do the following with the assistance of the free computer algebra system `Maxima`. I used the exact form of the metric tensor, and then approximated the resulting Christoffel symbols to first order in ϕ.

$$\Gamma^t{}_{t\alpha} = \frac{\partial\phi}{\partial x^\alpha}\frac{1}{1+2\phi} \approx \frac{\partial\phi}{\partial x^\alpha}(1-2\phi) \qquad \Gamma^\alpha{}_{t\alpha} = -\frac{\partial\phi}{\partial t}\frac{1}{1-2\phi} \approx -\frac{\partial\phi}{\partial t}(1+2\phi)$$

$$\Gamma^i{}_{tt} = \frac{\partial\phi}{\partial x^i}\frac{1}{1-2\phi} \approx \frac{\partial\phi}{\partial x^i}(1+2\phi) \qquad \Gamma^t{}_{ii} = -\frac{\partial\phi}{\partial t}\frac{1}{1+2\phi} \approx \frac{\partial\phi}{\partial t}(1-2\phi)$$

$$\Gamma^i{}_{jj} = -\Gamma^j{}_{ij} = -\Gamma^i{}_{ii} = \frac{\partial\phi}{\partial x^i}\frac{1}{1-2\phi} \approx \frac{\partial\phi}{\partial x^i}(1+2\phi)$$

5

(a) In the case of a perfect fluid, verify that the spatial components of Equation 7.6 reduce to

$$\dot{\boldsymbol{v}} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla}p/\rho + \boldsymbol{\nabla}\phi = 0$$

in the Newtonian limit and in the weak-field regime (the metric given by Equation 7.8).

$$\begin{aligned}
T^{\mu\nu} &= (\rho+p)U^\mu U^\nu + pg^{\mu\nu} \\
&\approx \rho U^\mu U^\nu + pg^{\mu\nu} \\
&\approx mU^\mu(nU^\nu) + pg^{\mu\nu} \\
T^{i\nu} &\approx mU^i(nU^\nu) + pg^{i\nu} \\
T^{i\nu}{}_{;\nu} &\approx m[U^i(nU^\nu)]_{;\nu} + [pg^{i\nu}]_{;\nu} = mnU^\nu U^i{}_{;\nu} + g^{i\nu}p_{;\nu} = 0 \\
\implies 0 &= U^\nu U^i{}_{;\nu} + g^{i\nu}p_{;\nu}/\rho \\
&= U^\nu(U^i{}_{,\nu} + U^\lambda\Gamma^i{}_{\lambda\nu}) + g^{i\nu}p_{,\nu}/\rho \\
&= U^0 U^i{}_{,0} + U^j U^i{}_{,j} + U^\nu U^\lambda \Gamma^i{}_{\lambda\nu} + g^{i\nu}p_{,\nu}/\rho \\
&= \gamma\frac{\mathrm{d}}{\mathrm{d}\tau}(\gamma v^i) + \gamma v^j(\gamma v^i)_{,j} + U^\nu U^\lambda\Gamma^i{}_{\lambda\nu} + g^{ii}p_{,i}/\rho \\
&\approx \frac{\mathrm{d}v^i}{\mathrm{d}\tau} + v^j v^i_{,j} + (U^0)^2\Gamma^i{}_{00} + (1-2\phi)p_{,i}/\rho \\
&\approx \frac{\mathrm{d}v^i}{\mathrm{d}\tau} + v^j v^i_{,j} + \phi_{,i} + p_{,i}/\rho
\end{aligned}$$

rewriting this in vector form, we get the original equation.

(b) Now look at the time-component instead of the spatial component.

$$\begin{aligned}
T^{0\nu} &= (\rho+p)U^0U^\nu + pg^{0\nu} \approx mU^0(nU^\nu) + pg^{0\nu} \\
T^{0\nu}{}_{;\nu} &= m[U^0(nU^\nu)]_{;\nu} + [pg^{0\nu}]_{;\nu} = mnU^\nu U^0{}_{;\nu} + g^{00}p_{,\nu} = 0 \\
\implies 0 &= U^\nu U^0{}_{;\nu} + g^{00}\dot{p}/\rho \\
&= U^\nu(U^0{}_{,\nu} + U^\lambda\Gamma^0{}_{\lambda\nu}) + g^{00}\dot{p}/\rho
\end{aligned}$$

$$
\begin{aligned}
&= U^0 U^0{}_{,0} + U^i U^0{}_{,i} + U^\nu U^\lambda \Gamma^0{}_{\lambda\nu} + g^{00}\dot{p}/\rho \\
&\approx \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}\tau} + \frac{1}{2}v^i\frac{\mathrm{d}v^2}{\mathrm{d}x^i} + U^\nu U^0 \Gamma^0{}_{0\nu} - (1+2\phi)\dot{p}/\rho \\
&\approx \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}\tau} + \frac{1}{2}v^i\frac{\mathrm{d}v^2}{\mathrm{d}x^i} + U^\nu U^0 \Gamma^0{}_{0\nu} - \dot{p}/\rho \\
&\approx \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}\tau} + \frac{1}{2}v^i\frac{\mathrm{d}v^2}{\mathrm{d}x^i} + U^\nu U^0 \phi_{,\nu} - \dot{p}/\rho \\
&\approx \frac{1}{2}\frac{\mathrm{d}v^2}{\mathrm{d}\tau} + \frac{1}{2}v^i\frac{\mathrm{d}v^2}{\mathrm{d}x^i} + \dot{\phi} + v^i\phi_{,i} - \dot{p}/\rho
\end{aligned}
$$

(c) A metric is static if there exist coordinates such that $\vec{e}_0$ is timelike, $g_{i0} = 0$, and $g_{\alpha\beta,0} = 0$. Show from Equation 7.6 that a static fluid (i.e. $U^i = 0$, $p_{,0} = 0$, etc) obeys the relativistic equation of hydrostatic equilibrium (Equation 7.40):

$$p_{,i} + (\rho + p)\left[\frac{1}{2}\ln(-g_{00})\right]_{,i} = 0.$$

We start by writing out Equation 7.6 as

$$
\begin{aligned}
T^{\mu\nu}{}_{;\nu} &= [(\rho+p)U^\mu U^\nu]_{;\nu} + [pg^{\mu\nu}]_{;\nu} = 0 \\
&= [(\rho+p)U^\mu]U^\nu{}_{;\nu} + [(\rho+p)U^\nu]U^\mu{}_{;\nu} + U^\nu U^\mu(\rho+p)_{,\nu} + g^{\mu\nu}p_{,\nu} = 0 \\
&= T^{00}{}_{;0} + T^{ij}{}_{;j} + T^{0i}{}_{;i} + T^{i0}{}_{;0} = 0 \\
T^{00}{}_{;0} &= [(\rho+p)U^0]U^0{}_{;0} + [(\rho+p)U^0]U^0{}_{;0} + U^0U^0(\rho+p)_{,0} + g^{00}p_{,0} \\
&= 2(\rho+p)U^0U^0{}_{;0} \\
&= 2(\rho+p)U^0[U^0{}_{,0} + U^\lambda\Gamma^0{}_{0\lambda}] \\
&= 2(\rho+p)[U^0]^2\Gamma^0{}_{00} = 0 \\
T^{ij}{}_{;j} &= [(\rho+p)U^i]U^j{}_{;j} + [(\rho+p)U^j]U^i{}_{;j} + U^jU^i(\rho+p)_{,j} + g^{ij}p_{,j} \\
&= g^{ij}p_{,j} \\
T^{0i}{}_{;i} &= [(\rho+p)U^0]U^i{}_{;i} + [(\rho+p)U^i]U^0{}_{;i} + U^iU^0(\rho+p)_{,i} + g^{0i}p_{,i} \\
&= [(\rho+p)U^0]U^i{}_{;i} = (\rho+p)U^0[U^i{}_{,i} + U^\lambda\Gamma^i{}_{i\lambda}] \\
&= (\rho+p)[U^0]^2\Gamma^i{}_{i0} \\
&= \frac{1}{2}[U^0]^2(\rho+p)g^{i\alpha}(g_{\alpha i,0} + g_{\alpha 0,i} - g_{0i,\alpha}) = 0 \\
T^{i0}{}_{;0} &= [(\rho+p)U^i]U^0{}_{;0} + [(\rho+p)U^0]U^i{}_{;0} + U^0U^i(\rho+p)_{,0} + g^{i0}p_{,0} \\
&= [(\rho+p)U^0]U^i{}_{;0} = (\rho+p)U^0[U^i{}_{,0} + U^0\Gamma^i{}_{00}] \\
&= \frac{1}{2}(\rho+p)[U^0]^2g^{i\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha}) = -\frac{1}{2}(\rho+p)[U^0]^2g^{ij}g_{00,j} \\
&= \frac{1}{2}(\rho+p)g^{ij}g_{00,j}/g_{00} = \frac{1}{2}(\rho+p)g^{ij}\ln(-g_{00})_{,j} \\
T^{\mu\nu}{}_{;\nu} &= g^{ij}p_{,j} + \frac{1}{2}(\rho+p)g^{ij}\ln(-g_{00})_{,j} = 0 \\
&= p_{,j} + (\rho+p)\left[\frac{1}{2}\ln(-g_{00})\right]_{,j} = 0
\end{aligned}
$$

(d) This suggests that there is a relationship between g_{00} and $\exp(2\phi)$ in the case of a static fluid in a Newtonian potential. Show that Equation 7.8 and Exercise 4 are consistent with this.

In the Newtonian limit, the previous equation is unchanged when replacing ng_{00} with $-\exp(2\phi)$, as $\ln\big(\exp(2\phi)\big)_{,i} = 2\phi_{,i}$, and

$$\begin{aligned} \ln(-g_{00})_{,i} &= \ln(1+2\phi)_{,i} = \frac{(1+2\phi)_{,i}}{1+2\phi} \\ &= 2\phi_{,i}(1+2\phi)^{-1} \approx 2\phi_{,i}(1-2\phi) \approx 2\phi_{,i}. \end{aligned}$$

I'm not really sure how to relate this to Exercise 4, as it relates $\phi_{,\alpha}$ to four-momentum, while this relates it to pressure and density.

7 Consider the (i) Minkowski, (ii) Schwarzschild, (iii) Kerr, and (iv) Robertson–Walker metrics.

(a) Find the conserved components p_α of a the four-momentum of a particle in free-fall.

For this I will use Equation 7.29:

$$m\frac{\mathrm{d}p_\beta}{\mathrm{d}\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha.$$

What this tells us is that if $g_{\alpha\beta}$ is independent of x^μ, then p_μ is constant along the trajectory.

For (i), the metric is independent of all coordinates (t, x, y, z), and so all p_α are conserved.

For (ii), the metric depends on coordinates r and θ, but not t and ϕ, so only p_t and p_ϕ are conserved.

For (iii) we have the same dependencies as (ii).

For (iv) there is an additional time dependence, and so only p_ϕ is conserved.

(b) Use the metric for a flat spacetime in spherical polar coordinates to argue that the Schwarzschild and Robertson–Walker metrics are spherically symmetric.

Our metric in (i) can be expressed in spherical polars as

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

The Schwarzschild metric can be obtained from this by multiplying $\mathrm{d}t^2$ by $(1-2M/r)$, and dividing $\mathrm{d}r^2$ by it. This newly introduced term only introduces a new radial dependence (the r^{-1} term), not an angular one, so it retains spherical symmetry.

The Robertson–Walker metric can be obtained by dividing $\mathrm{d}r^2$ by $(1-kr^2)$, and then multiplying everything *except* $\mathrm{d}t^2$ by $R^2(t)$. Again, the $(1-kr^2)$ term only introduces a radial dependence in its r^2 term, and for a given time t, $R^2(t)$ is a constant, so spherical symmetry is retained.

(c) For (i') and (ii)–(iv), a geodesic which at one point has $\theta = \pi/2$ and $p^\theta = 0$ (i.e. tangent to the equatorial plane) conserves these quantities. For (i'), (ii), and (iii),use $\vec{p}\cdot\vec{p} = -m^2$ to find p^r as a function of m, other conserved quantities, and known functions of position.

(i')

$$\vec{p}\cdot\vec{p} = g_{\alpha\beta}p^\alpha p^\beta = g_{\alpha\alpha}(p^\alpha)^2 = g_{tt}(p^t)^2 + g_{rr}(p^r)^2 + g_{\theta\theta}(\cancelto{0}{p^\theta})^2 + g_{\phi\phi}(p^\phi)^2$$

$$= -(p^t)^2 + (p^r)^2 + r^2 \cancelto{1}{\sin^2(\theta)}(p^\phi)^2 = -m^2$$

$$\implies (p^r)^2 = (p^t)^2 - r^2(p^\phi)^2 - m^2 = g^{tt}(p_t)^2 - r^2 g^{\phi\phi}(p_\phi)^2 - m^2 = -(p^t)^2 - (p^\phi)^2 - m^2$$

$$\implies p^r = \pm\sqrt{-[(p^t)^2 + (p^\phi)^2 + m^2]}$$

(ii)

$$\begin{aligned}
\vec{p}\cdot\vec{p} &= g_{tt}(p^t)^2 + g_{rr}(p^r)^2 + g_{\phi\phi}(p^\phi)^2 \\
&= -(1-2M/r)(p^t)^2 + (1-2M/r)^{-1}(p^r)^2 + r^2 \cancelto{1}{\sin^2\theta}(p^\phi)^2 = -m^2 \\
\implies (p^r)^2 &= (1-2M/r)[(1-2M/r)(p^t)^2 - r^2(p^\phi)^2 - m^2] \\
&= -(1-2M/r)[(1-2M/r)(p_t)^2 + (p_\phi)^2 + m^2]
\end{aligned}$$

(iii) This metric gets a bit messy, so I will keep things more abstract. First, I will simplify the metric, utilizing the fact that $\theta = \pi/2$.

$$\mathrm{d}s^2 = -\frac{\Delta - a^2}{r^2}\,\mathrm{d}t^2 - 2\frac{2Ma}{r}\,\mathrm{d}t\,\mathrm{d}\phi + \frac{(r^2+a^2)^2 - a^2\Delta}{r^2}\mathrm{d}\phi^2 + \frac{r^2}{\Delta}\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$$

$$g_{tt} = -\frac{\Delta - a^2}{r^2}; \quad g_{rr} = \frac{r^2}{\Delta}; \quad g_{\theta\theta} = r^2; \quad g_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta}{r^2}; \quad g_{t\phi} = -\frac{2Ma}{r},$$

$$\lambda \equiv a^6 - 2(D - r^2)a^4 + (r^4 - 4M^2r^2 - 2Dr^2 + D^2)a^2 - Dr^4$$

$$g^{tt} = r^2(a^4 - (D - 2r^2)a^2 + r^4)/\lambda; \quad g^{rr} = \frac{D}{r^2}; \quad g^{\theta\theta} = \frac{1}{r^2}; \quad g^{\phi\phi} = r^2(a^2 - D)/\lambda; \quad g^{t\phi} = 2aMr^3/\lambda,$$

$$\begin{aligned}
\vec{p}\cdot\vec{p} &= g_{tt}(p^t)^2 + g_{rr}(p^r)^2 + g_{\phi\phi}(p^\phi)^2 + 2g_{t\phi}(p^t p^\phi) = -m^2 \\
p^r &= \pm\sqrt{-g^{rr}[g_{tt}(p^t)^2 + g_{\phi\phi}(p^\phi)^2 + 2g_{t\phi}(p^t p^\phi) + m^2]} \\
p^t &= g^{t\alpha}p_\alpha = g^{\phi\phi}p_t + g^{t\phi}p_\phi \\
p^\phi &= g^{\phi\alpha}p_\alpha = g^{\phi\phi}p_\phi + g^{t\phi}p_t
\end{aligned}$$

(d)

When $k = 0$, the line element and metric become

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + R^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]$$

$$g_{tt} = -1; \quad g_{rr} = R^2(t); \quad g_{\theta\theta} = R^2(t)r^2; \quad g_{\phi\phi} = R^2(t)r^2\sin^2\theta.$$

Equation 7.29 with $\beta = r$ then becomes

$$m\frac{\mathrm{d}p_r}{\mathrm{d}\tau} = \frac{1}{2}g_{\nu\alpha,r}p^\nu p^\alpha = \frac{1}{2}[g_{tt,r}(p^t)^2 + g_{rr,r}(p^r)^2].$$

Since $g_{tt,r} = g_{rr,r} = 0$, the RHS becomes zero, and so

$$m\frac{\mathrm{d}p_r}{\mathrm{d}\tau} = 0 \implies p_r \text{ is conserved.}$$

8 For a coordinate system where $g_{\alpha\beta,\mu} = 0$:

(a) Show that ${T^{\nu}}_{\mu;\nu} = 0$ becomes

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}{T^{\nu}}_{\mu})_{\nu} = 0.$$

For this, I will make mathematicians cry, and go from the *solution* backwards to the starting point. So I expand the final expression, first using the Leibniz rule:

$${T^{\nu}}_{\mu,\nu} + \frac{(\sqrt{-g})_{\nu}}{\sqrt{-g}}{T^{\nu}}_{\mu} = 0,$$

and then using Equation 6.40:

$${T^{\nu}}_{\mu,\nu} + {\Gamma^{\alpha}}_{\alpha\nu} = 0.$$

Just pretend I did that backwards. Next I expand ${T^{\nu}}_{\mu;\nu}$, to show that the above expression makes it zero.

$$\begin{aligned}
{T^{\nu}}_{\mu;\nu} &= {T^{\nu}}_{\mu,\nu} + {T^{\alpha}}_{\mu}{\Gamma^{\nu}}_{\alpha\nu} - {T^{\nu}}_{\alpha}{\Gamma^{\alpha}}_{\mu\nu} \\
&= {T^{\nu}}_{\mu,\nu} + {T^{\nu}}_{\mu}{\Gamma^{\alpha}}_{\nu\alpha} - {T^{\nu}}_{\alpha}{\Gamma^{\alpha}}_{\mu\nu}.
\end{aligned}$$

Note that the positive terms are just the expression from before, which we showed was zero, so we're left with

$${T^{\nu}}_{\mu;\nu} = {T^{\nu}}_{\alpha}{\Gamma^{\alpha}}_{\mu\nu}.$$

Now we expand this

$$\begin{aligned}
{T^{\nu}}_{\mu;\nu} &= -\frac{1}{2}{T^{\nu}}_{\alpha}g^{\alpha\beta}(g_{\beta\mu,\nu} + \cancel{g_{\beta\nu,\mu}} - g_{\mu\nu,\beta}) \\
&= -\frac{1}{2}T^{\nu\beta}(g_{\beta\mu,\nu} - g_{\mu\nu,\beta}) = -\frac{1}{2}T^{(\nu\beta)}A_{[\nu\beta]\mu} = 0.
\end{aligned}$$

(b) Suppose $T^{\alpha\beta}$ is zero except in a bounded region of the space-like hypersurface $x^0 =$ constant. Show that Equation 7.41 implies that

$$\int_{x^0=\mathrm{const}} {T^{\nu}}_{\mu}\sqrt{-g}n_{\nu}\,\mathrm{d}^3x$$

does not depend on x^0, so long as n_ν is the unit normal to the hypersurface.

Using Equation 7.41 and the differential in Equation 6.18, we take the integral

$$\int \frac{1}{\sqrt{-g}}(\sqrt{-g}{T^{\nu}}_{\mu})_{\nu}\sqrt{-g}\,\mathrm{d}^4x = \int (\sqrt{-g}{T^{\nu}}_{\mu})_{,\nu}\,\mathrm{d}^4x\,\mathrm{d}^4x\,.$$

Now we use Equation 6.44:

$$\begin{aligned}
\int (\sqrt{-g}{T^{\nu}}_{\mu})_{,\nu}\,\mathrm{d}^4x\,\mathrm{d}^4x &= \oint \sqrt{-g}n_{\nu}{T^{\nu}}_{\mu}\,\mathrm{d}^3S \\
&= \int_{x^0=\mathrm{const}} \sqrt{-g}n_{\nu}{T^{\nu}}_{\mu}\,\mathrm{d}^3x
\end{aligned}$$

(c) Now consider flat Minkowski space with a global inertial frame in spherical polar coordinates. Show that, from part (b), we have

$$J = \int_{t=\mathrm{const}} {T^{0}}_{\phi}\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\mathrm{d}\phi,$$

which is independent of t. This is the system's total angular momentum.

Since we are in flat Minkowski space, the unit-normal one form has components $\tilde{n} \to (1,0,0,0)$, so only the $T^0{}_\mu$ term is retained. We also have $x^0 \to t$, so we can write the expression from (b) as

$$\int_{t=\text{const}} \sqrt{-g} T^0{}_\mu \, \mathrm{d}^3x \, .$$

We also know that $\sqrt{-g}\,\mathrm{d}^3x$ in spherical polars is $r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \mathrm{d}\phi$, so we can write this as

$$\int_{t=\text{const}} T^0{}_\mu \, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \mathrm{d}\phi.$$

Taking the ϕ component of $T^0{}_\mu$, we get something which we call J:

$$J = \int_{t=\text{const}} T^0{}_\phi \, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \mathrm{d}\phi.$$

(d) Now express the previous integral in terms of the components of $T^{\alpha\beta}$ on the Cartesian basis, ultimately arriving at

$$J = \int (xT^{y0} - yT^{x0}) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$\begin{aligned}
J &= \int_{t=\text{const}} T^0{}_\phi \, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \mathrm{d}\phi \\
&= \int_{t=\text{const}} \Lambda^\alpha{}_\phi T^0{}_\alpha \, r^2 \sin\theta \, \mathrm{d}^3x \\
&= \int_{t=\text{const}} (\Lambda^x{}_\phi T^0{}_x + \Lambda^y{}_\phi T^0{}_y + \Lambda^z{}_\phi T^0{}_z) \, \mathrm{d}^3x \\
&= \int_{t=\text{const}} ((-r\sin\theta\sin\phi) T^0{}_x + (r\sin\theta\cos\phi) T^0{}_y + (0) T^0{}_z) \, \mathrm{d}^3x \\
&= \int_{t=\text{const}} (xT^0{}_y - yT^0{}_x) \, \mathrm{d}^3x \\
&= \int_{t=\text{const}} (\eta_{yy} x T^{0y} - \eta_{xx} y T^{0x}) \, \mathrm{d}^3x \\
&= \int_{t=\text{const}} (xT^{0y} - yT^{0x}) \, \mathrm{d}^3x
\end{aligned}$$

10

(a) Show that if the vector field ξ^α satisfies Killing's equation,

$$\boldsymbol{\nabla}_\alpha \xi_\beta + \boldsymbol{\nabla}_\beta \xi_\alpha = 0,$$

then $p^\alpha \xi_\alpha$ is constant along a geodesic.

If $p^\alpha \xi_\alpha$ is constant along a geodesic, then $p^\alpha \xi_{\alpha;\beta} = 0$, so we simply have to show that this follows from Killing's equation.

Killing's equation can be rewritten as

$$\xi_{\beta;\alpha} + \xi_{\alpha;\beta} = 0 \implies \xi_{\beta;\alpha} = -\xi_{\alpha;\beta}.$$

Now we combine this with the geodesic equation,

$$p^\alpha \xi_{\beta;\alpha} = -p^\alpha \xi_{\alpha;\beta} = 0.$$

And there we have it!

(b) Find ten Killing fields for Minkowski spacetime.

Since the basis vectors in Minkowski spacetime are all constant, $\boldsymbol{\nabla}_\beta \vec{e}_\alpha = 0$, and so we get four from $\vec{e}_t$, $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$. According to part (c), we get a Killing field from any *constant* linear combination of these four, and so from that we may create an infinity of Killing fields. Schutz's solutions manual also lists expressions such as $x\vec{e}_t - t\vec{e}_x$ as Killing fields, which are linear combinations, but the coefficients are non-constant. I give an attempted derivation below, although at the very last step it turns out not to work, and I pretend it does anyway. I claim that the general form of Schutz's expressions is: $x^\alpha \vec{e}_\beta - x^\beta \vec{e}_\alpha$.

$$\begin{aligned}
\boldsymbol{\nabla}_\alpha(x^\alpha \vec{e}_\beta) - \boldsymbol{\nabla}_\alpha(x^\beta \vec{e}_\alpha) + \boldsymbol{\nabla}_\beta(x^\alpha \vec{e}_\beta) - \boldsymbol{\nabla}_\beta(x^\beta \vec{e}_\alpha) &= x^\alpha{}_{;\alpha}\vec{e}_\beta - x^\beta{}_{;\alpha}\vec{e}_\alpha + x^\alpha{}_{;\beta}\vec{e}_\beta - x^\beta{}_{;\beta}\vec{e}_\alpha \\
&= \vec{e}_\beta - \vec{e}_\alpha - x^\beta{}_{,\alpha}\vec{e}_\alpha + x^\alpha{}_{,\beta}\vec{e}_\beta \\
&= \vec{e}_\beta - \vec{e}_\alpha - \Lambda^\beta{}_\alpha \vec{e}_\alpha + \Lambda^\alpha{}_\beta \vec{e}_\beta \\
&\text{(magnets at work here)} \\
&= \vec{e}_\beta - \vec{e}_\alpha - \vec{e}_\beta + \vec{e}_\alpha = 0
\end{aligned}$$

(c) Prove that any *constant* linear combination of two Killing fields $\vec{\xi}$ and $\vec{\eta}$ is itself a Killing field.

$$\begin{aligned}
&\boldsymbol{\nabla}_\mu \xi_\nu + \boldsymbol{\nabla}_\nu \xi_\mu = 0 \\
&\boldsymbol{\nabla}_\mu \eta_\nu + \boldsymbol{\nabla}_\nu \eta_\mu = 0 \\
&\boldsymbol{\nabla}_\mu(\alpha\xi_\nu + \beta\eta_\nu) + \boldsymbol{\nabla}_\nu(\alpha\xi_\mu + \beta\eta_\mu) \\
=&\alpha\boldsymbol{\nabla}_\mu\xi_\nu + \beta\boldsymbol{\nabla}_\mu\eta_\nu + \alpha\boldsymbol{\nabla}_\nu\xi_\mu + \beta\boldsymbol{\nabla}_\nu\eta_\mu \\
=&\alpha(\boldsymbol{\nabla}_\mu\xi_\nu + \boldsymbol{\nabla}_\nu\xi_\mu) + \beta(\boldsymbol{\nabla}_\mu\eta_\nu + \boldsymbol{\nabla}_\nu\eta_\mu) = 0
\end{aligned}$$

(d) Show that the Lorentz transforms of the fields in (b) are also Killing fields.

Applying a Lorentz transform $\Lambda^\mu{}_\nu$ we get the expression $\Lambda^\mu{}_\nu\big(x^\alpha \vec{e}_\beta - x^\beta \vec{e}_\alpha\big)$.

$$\begin{aligned}
&\boldsymbol{\nabla}_\alpha[\Lambda^\mu{}_\nu(x^\alpha \vec{e}_\beta - x^\beta \vec{e}_\alpha)] + \boldsymbol{\nabla}_\beta[\Lambda^\mu{}_\nu(x^\beta \vec{e}_\alpha - x^\alpha \vec{e}_\beta)] \\
=&\Lambda^\mu{}_{\nu;\alpha}[(x^\alpha \vec{e}_\beta - x^\beta \vec{e}_\alpha) + (x^\beta \vec{e}_\alpha - x^\alpha \vec{e}_\beta)] \\
=&\Lambda^\mu{}_{\nu;\alpha}[x^\alpha \vec{e}_\beta - x^\alpha \vec{e}_\beta + x^\beta \vec{e}_\alpha - x^\beta \vec{e}_\alpha] = 0
\end{aligned}$$

(e) Use the results in Exercise 7(a) to find Killing vectors for the non-Minkowski metrics listed in (ii)–(iv).

(ii) Since the conserved quantities are p_t and p_ϕ, then the Killing fields are any constant linear combinations or Lorentz transforms of $\vec{e}_t$, $\vec{e}_\phi$, and $\phi\vec{e}_t - t\vec{e}_\phi$.

(iii) Same as (ii).

(iv) Only p_ϕ is conserved, so any constant multiple of $\vec{e}_\phi$ is a Killing field.

Chapter 8

The Einstein field equations

8.6 Exercises

3

(a) Calculate in geometrized units:

(i) the Newtonian potential of the Sun at its surface

$$\phi = -GM_\odot/R_\odot \approx -1.476 \times 10^3\,\mathrm{m}/6.960 \times 10^8\,\mathrm{m} \approx -2.12 \times 10^{-6}$$

(ii) the Newtonian potential of the Sun at the radius of Earth's orbit

$$\phi = -GM_\odot/1\,\mathrm{AU} \approx -1.476 \times 10^3\,\mathrm{m}/1.496 \times 10^{11}\,\mathrm{m} \approx -9.866 \times 10^{-9}$$

(iii) the Newtonian potential of the Earth at its surface

$$\phi = -GM_\oplus/R_\oplus \approx -4.434 \times 10^{-3}\,\mathrm{m}/6.371 \times 10^6\,\mathrm{m} \approx -9.660 \times 10^{-10}$$

(iv) the Earth's orbital velocity

Here I use the result from part (c), and find that

$$v = \sqrt{-\phi} \approx 9.933 \times 10^{-5}$$

(b) If the potential due to the Sun at Earth's orbital radius is greater than the Earth's potential at its surface (as is shown above), then why do we feel the Earth's gravity more than the Sun's?

We don't feel the potential directly, we feel the gravitational acceleration it produces. Acceleration is obtained from the potential via $\mathbf{a} = -\boldsymbol{\nabla}\phi$, and in the case of a circular orbit in a Newtonian potential:

$$a = -\boldsymbol{\nabla}\phi = -\frac{\partial}{\partial r}(-GM/r) = -Gm/r^2 = \phi/r.$$

So in the two cases mentioned, we need to divide by the radius once more, to obtain the acceleration.

$$a_\odot = \phi_\odot/1\,\mathrm{AU} \approx -6.595 \times 10^{-20}\,\mathrm{m}^{-1}$$
$$a_\oplus = \phi_\oplus/R_\oplus \approx -1.092 \times 10^{-16}\,\mathrm{m}^{-1}$$

As you can see, the acceleration due to the Earth is greater by a factor of 10^4.

(c) Show that a circular orbit in a Newtonian potential has an orbital velocity $v^2 = -\phi$.

We saw above that $a = \phi/r$, and we also know that centripetal acceleration is given by $a = -v^2/r$. Equating the two we get $v^2 = -\phi$.

8

(a) Show that $R^\alpha{}_{\beta\mu\nu} = \eta^{\alpha\sigma}R_{\alpha\beta\mu\nu} + \mathcal{O}([h_{\alpha\beta}]^2)$.

$$R^\alpha{}_{\beta\mu\nu} = g^{\alpha\sigma}R_{\sigma\beta\mu\nu} = (\eta^{\alpha\sigma} + h^{\alpha\sigma})R_{\sigma\beta\mu\nu} = \eta^{\alpha\sigma}R_{\sigma\beta\mu\nu} + h^{\alpha\sigma}R_{\sigma\beta\mu\nu}$$
$$h^{\alpha\sigma}R_{\sigma\beta\mu\nu} = \frac{1}{2}h^{\alpha\sigma}(h_{\sigma\nu,\beta\mu} + h_{\beta\mu,\sigma\nu} - h_{\sigma\mu,\beta\nu} - h_{\beta\nu,\sigma\mu}) = \mathcal{O}([h_{\alpha\beta}]^2)$$

(b) Find $R_{\alpha\beta}$ to first order in $h_{\mu\nu}$.

$$R^\alpha{}_{\beta\mu\nu} \approx \eta^{\alpha\sigma}R_{\sigma\beta\mu\nu}$$
$$\delta^\mu{}_\alpha R^\alpha{}_{\beta\mu\nu} \approx R_{\beta\nu} \approx \delta^\mu{}_\alpha\eta^{\alpha\sigma}R_{\sigma\beta\mu\nu} \approx \eta^{\mu\sigma}R_{\sigma\beta\mu\nu}$$

(c) Show that $g_{\alpha\beta}R = \eta_{\alpha\beta}\eta^{\mu\nu}R_{\mu\nu} + \mathcal{O}([h_{\alpha\beta}]^2)$.

$$R = g^{\mu\nu}R_{\mu\nu} = (\eta^{\mu\nu} + h^{\mu\nu})R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu} + \eta^{\mu\gamma}\eta^{\nu\lambda}R_{\mu\nu} = \eta^{\mu\nu}R_{\mu\nu} + \mathcal{O}([h_{\alpha\beta}]^2)$$
$$g_{\alpha\beta}R = g_{\alpha\beta}\eta^{\mu\nu}R_{\mu\nu} + \mathcal{O}([h_{\alpha\beta}]^2) = (\eta_{\alpha\beta} + h_{\alpha\beta})\eta^{\mu\nu}R_{\mu\nu} + \mathcal{O}([h_{\alpha\beta}]^2) = \eta_{\alpha\beta}\eta^{\mu\nu}R_{\mu\nu} + \mathcal{O}([h_{\alpha\beta}]^2)$$

(d) Use this to show that $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R$.

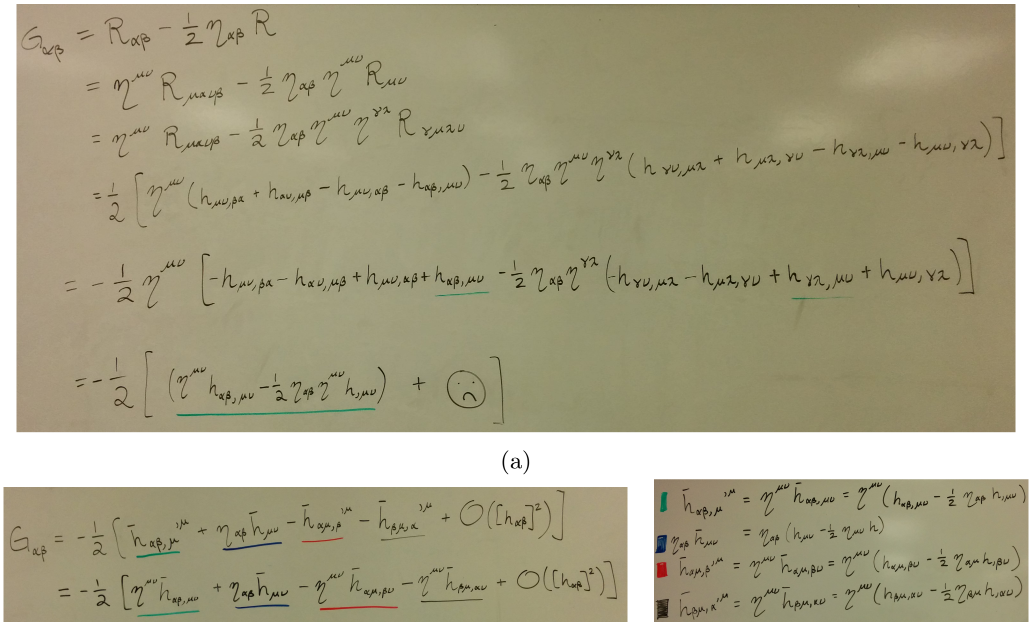
$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = R_{\alpha\beta} - \frac{1}{2}(\eta_{\alpha\beta}\eta^{\mu\nu}R_{\mu\nu}) = R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}R$$

(e) Now use this to simplify the calculation of Equation 8.32.

I got stuck here. I began by expanding the expression in (d) using the results from previous sections, shown in Figure 8.1a. Then I expanded Equation 8.32, to get it in a more similar form, in Figures 8.1b and 8.1c. I did this with the hope of matching terms in the two equations, but was only able to match one. I believe something that would help me get further is Equation 8.33, $\bar{h}^{\mu\nu}{}_{,\nu} = 0$.

9

(a)



(a) (b) (c)

Figure 8.1: Incomplete solution to Problem 8.8 (e)

I start by making a slight rewrite of Equation 8.32, changing the second $\bar{h}$ term.

$$\eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} = \eta_{\alpha\beta}\eta^{\mu\alpha}\bar{h}_{\mu\nu,\alpha}{}^{,\nu} = \bar{h}_{\beta\nu,\alpha}{}^{,\nu}$$

So now the Einstein tensor can be written as

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \bar{h}_{\beta\nu,\alpha}{}^{,\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu} + \mathcal{O}([h_{\alpha\beta}]^2)].$$

For G_{00} we then have

$$\begin{aligned}
G_{00} &= -\frac{1}{2}[\bar{h}_{00,\mu}{}^{,\mu} + \bar{h}_{0\nu,0}{}^{,\nu} - \bar{h}_{0\mu,0}{}^{,\mu} - \bar{h}_{0\mu,0}{}^{,\mu} + \mathcal{O}([h_{00}]^2)] \\
&= -\frac{1}{2}[\bar{h}_{00,\mu}{}^{,\mu} - \bar{h}_{0\mu,0}{}^{,\mu} + \mathcal{O}([h_{00}]^2)] \\
&= -\frac{1}{2}[(\bar{h}_{00,0}{}^{,0} + \bar{h}_{00,i}{}^{,i}) - (\bar{h}_{00,0}{}^{,0} + \bar{h}_{0i,0}{}^{,i}) + \mathcal{O}([h_{00}]^2)] \\
&= -\frac{1}{2}[\bar{h}_{00,i}{}^{,i} - \bar{h}_{0i,0}{}^{,i} + \mathcal{O}([h_{00}]^2)],
\end{aligned}$$

which contains no second time derivatives. For G_{0i} I encountered a problem:

$$\begin{aligned}
G_{0i} &= -\frac{1}{2}[\bar{h}_{0i,\mu}{}^{,\mu} + \bar{h}_{i\nu,0}{}^{,\nu} - \bar{h}_{0\mu,i}{}^{,\mu} - \bar{h}_{i\mu,0}{}^{,\mu} + \mathcal{O}([h_{0i}]^2)] \\
&= -\frac{1}{2}[\bar{h}_{0i,\mu}{}^{,\mu} - \bar{h}_{0\mu,i}{}^{,\mu} + \mathcal{O}([h_{0i}]^2)] \\
&= -\frac{1}{2}[(\bar{h}_{0i,0}{}^{,0} + \bar{h}_{0i,j}{}^{,j}) - \bar{h}_{0\mu,i}{}^{,\mu} + \mathcal{O}([h_{0i}]^2)],
\end{aligned}$$

which retains a second time derivative in the $\bar{h}_{0i,0}{}^{,0}$ term.

(b)

According to Schutz's solution it is not a contradiction, due in part to Equation 8.33. I don't fully understand the reason, though.

11 Write the gauge transformation and Lorentz gauge condition in four-tensor notation for Maxwell's equations. Draw an analogy with linearized gravity.

First we rewrite $\phi \to \phi - \partial f/\partial t$ as $-A_0 \to -A_0 - f_{,0}$, and cancelling the negatives we get $A_0 \to A_0 + f_{,0}$. Combining this with $A_i \to A_i + f_{,i}$, it is obvious that the gauge transformation generalizes to $A_\alpha \to A_\alpha + f_{,\alpha}$. The Lorentz gauge condition is just slightly less obvious. We start by noting that $A_0 = -\phi$, and therefore (in Minkowski space) $A^0 = \eta^{0\mu} A_\mu = g^{00} A_0 = (-1)(-\phi) = \phi$. Then the Lorentz gauge condition becomes ${A^0}_{,0} + {A^i}_{,i} = {A^\alpha}_{,\alpha} = 0$.

13 Give a physical justification for $\left|T^{00}\right| \gg \left|T^{0i}\right| \gg \left|T^{ij}\right|$ in a Newtonian system.

The first inequality is easy to see. $T^{00} = E/V = p^0/V$, and $T^{0i} = p^i/V$. In the Newtonian limit, $\left|p^0\right| \gg \left|p^i\right|$, and so it follows that $\left|T^{00}\right| \gg \left|T^{0i}\right|$.

The second inequality is less obvious. In the Newtonian limit, forces must be relatively small, or else objects would be accelerated to relativistic speeds. By this argument, the stresses must also be relatively small, and so $T^{0i} \gg T^{ij}$.

17

(a) First I need to convert the orbital period into meters.

$$T = 200\,\text{days} \times \frac{24\,\text{hours}}{1\,\text{day}} \times \frac{3600\,\text{seconds}}{1\,\text{hour}} \times c \approx 5.18 \times 10^{15}\,\text{m}$$

Then I use the potential to find the speed, which I relate to the circumference and orbital period, and solve for the mass.

$$\phi = -GM/r$$
$$v^2 = -\phi$$

$$\begin{aligned} M = v^2 r/G = C^3/(2\pi T^2 G) &\approx (6 \times 10^{11}\,\text{m})^3/(2\pi(5.18 \times 10^{15}\,\text{m})^2 G) \\ &\approx 1.281 \times 10^3\,\text{m} \times \frac{1 M_\odot}{1.476 \times 10^3\,\text{m}} \approx 0.868 M_\odot \end{aligned}$$

(b)

Using the above formula, I get a distribution of mass estimates, shown in Figure 8.2. Closer to the black hole, the Newtonian approximation breaks down, and the "effective mass" blows up. Far from the black hole, we can see that the effective mass is in agreement for all of the sattelites, and so the Newtonian approximation is working again. Thus, I use the furthest sattelite to find that the black hole's mass is $68 M_\odot$.

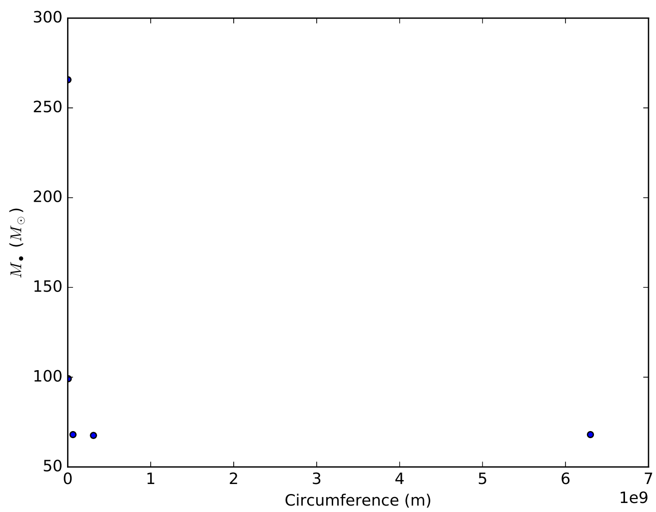


Figure 8.2: Black hole mass estimates in Problem 8.17, as a function of sattelite circumference.