Quantum Mechanics – Chapter 3

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Hilbert Space

Linear Algebra

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- the state of a system is represented by its wave function
- observables are represented by operators
- wave functions satisfy the defining conditions for *abstract vectors*
- operators act on them as *linear transformations*

Vectors

- in an N-dimensional space, a vector $|\alpha\rangle$ may be represented by the N-tuple of its components, $\{a_n\}$, with respect to a specified orthonormal basis

$$|\alpha\rangle \to \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

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- the “vectors” in quantum mechanics are typically functions, existing in *infinite*-dimensional spaces
  - the N-tuple notation used to represent finite-dimensional vectors becomes problematic

Bra–ket Notation

- the inner product of two vectors, $|\alpha\rangle$ and $|\beta\rangle$, is a generalization of the dot product, and is denoted $\langle\alpha|\beta\rangle$

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- here, $\langle\alpha|$ is called the "bra", and $|\beta\rangle$ is called the "ket"
- when α and β are functions on the interval (a, b), the inner product is given by the familiar integral

$$\langle\alpha|\beta\rangle = \int_a^b \alpha(x)^* \beta(x)\,\mathrm{d}x$$

Hilbert Space

- a wave function must be normalized, i.e.

$$\langle \Psi | \Psi \rangle = \int_{-\infty}^{\infty} \Psi^* \Psi \, \mathrm{d}x = \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x = 1$$

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- wave functions live in Hilbert space

Schwarz Inequality

- the integral Schwarz inequality states

$$\left| \int_a^b f(x)^* g(x) \, \mathrm{d}x \right| \leq \sqrt{\int_a^b |f(x)|^2 \, \mathrm{d}x \int_a^b |g(x)|^2 \, \mathrm{d}x}$$

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- the left side, or the magnitude of the inner product of our two functions, must be finite as well

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- a set of functions $\{f_n\}$ is orthonormal if $\langle f_m|f_n\rangle = \delta_{mn}$

Complete Functions

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- the stationary states for the harmonic oscillator form a complete orthonormal set on the interval $(-\infty, \infty)$

Observables

Hermitian Operators

- the expectation value of an operator $Q(x, p)$ can be expressed as

$$\langle Q \rangle = \int \Psi^* \hat{Q} \Psi \, \mathrm{d}x = \left\langle \Psi \middle| \hat{Q} \Psi \right\rangle$$

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- such operators are called *hermitian*

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- a *determinate* state, for a given observable Q, is a special case, in which each observation gives the same value, q

Determinate States

- the standard deviation of an observable Q, in a determinate state would be zero

$$\begin{aligned}\sigma^2 &= \left\langle (\hat{Q} - \langle Q \rangle)^2 \right\rangle \\ &= \left\langle \Psi \middle| (\hat{Q} - q)^2 \Psi \right\rangle \\ &= \left\langle (\hat{Q} - q)\Psi \middle| (\hat{Q} - q)\Psi \right\rangle = 0\end{aligned}$$

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- all determinate states are eigenfunctions of $\hat{Q}$

Spectrum

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- in the case where two or more linearly independent eigenfunctions share an eigenvalue, the spectrum is said to be *degenerate*

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- including the time dependence $\varphi(t)$ to make it Ψ does not change the fact that it is an eigenfunction of $\hat{H}$

Thank you!