Griffiths Chapter 2

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Problem 2.4

Calculate $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$, σ_x, and σ_p, for the nth stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

The nth stationary state is given by

$$\psi_n(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right)$$

It follows that

$$\begin{aligned}
\langle x \rangle_n &= \int_{-\infty}^{\infty} x |\psi_n(x)|^2 \, \mathrm{d}x = \frac{2}{\ell} \int_0^{\ell} x \sin^2\left(\frac{n\pi x}{\ell}\right) \mathrm{d}x = \frac{\ell}{2} \\
\langle x^2 \rangle_n &= \int_0^{\ell} x^2 |\psi_n(x)|^2 \, \mathrm{d}x = \frac{\ell^2}{6}\left[2 - \frac{3}{(n\pi)^2}\right] \\
\langle p \rangle_n &= \int_0^{\ell} \psi_n^* \left(\frac{\hbar}{\imath} \frac{\partial}{\partial x}\right) \psi_n \, \mathrm{d}x = -\frac{2\imath\hbar}{\ell} \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \frac{\partial}{\partial x} \sin\left(\frac{n\pi x}{\ell}\right) \mathrm{d}x \\
&= -\frac{2\imath\hbar}{n\pi} \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi x}{\ell}\right) \mathrm{d}x = 0 \\
\langle p^2 \rangle_n &= \int_0^{\ell} \psi_n^* \left(\frac{\hbar}{\imath} \frac{\partial}{\partial x}\right)^2 \psi_n \, \mathrm{d}x = -\hbar^2 \int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi x}{\ell}\right) = \\
&= \frac{2}{\ell}\left(\frac{\pi\hbar n}{\ell}\right)^2 \int_0^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right) \mathrm{d}x = \left(\frac{n\pi\hbar}{\ell}\right)^2 \\
\sigma_x &= \sqrt{\langle x^2 \rangle_n - \langle x \rangle_n^2} = \sqrt{\frac{\ell^2}{6}\left[2 - \frac{3}{(n\pi)^2}\right] - \frac{\ell^2}{2^2}} = \frac{\ell}{2\sqrt{3}} \sqrt{1 - \frac{6}{(n\pi)^2}} \\
\sigma_p &= \langle p^2 \rangle_n = \left(\frac{n\pi\hbar}{\ell}\right) \\
\sigma_x \sigma_p &= \frac{\ell}{2\sqrt{3}} \sqrt{1 - \frac{6}{(n\pi)^2}} \left(\frac{n\pi\hbar}{\ell}\right) = \hbar \sqrt{\frac{n^2\pi^2 - 6}{12}}
\end{aligned}$$

$$\begin{aligned}
&\text{For } n = 1 : \sigma_x \sigma_p = \hbar \sqrt{\frac{\pi^2 - 6}{12}} \approx 0.5678\hbar > \frac{\hbar}{2} \\
&\text{For } n \to \infty : \sigma_x \sigma_p \to \infty
\end{aligned}$$

Problem 2.5

A particle in the infinite square well has its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A\left[\psi_1(x) + \psi_2(x)\right]$$

a. Normalize $\Psi(x,0)$.

$$1 = \langle\Psi|\Psi\rangle = A^2 \left\langle(\psi_1+\psi_2)\middle|(\psi_1+\psi_2)\right\rangle = A^2 \left(\langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle\right) = 2A^2 \implies |A| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Psi(x,0) = \frac{\sqrt{2}}{2}\left[\psi_1(x) + \psi_2(x)\right]$$

b. Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega \equiv \pi^2\hbar/2m\ell^2$.

$$\begin{aligned}
\exp\left(-\imath E_n t/\hbar\right) &= \exp\left(-n^2\omega t\right)\\
\Psi(x,t) &= \frac{\sqrt{2}}{2}\left[\psi_1(x)\exp(-\omega t) + \psi_2(x)\exp(-4\omega t)\right]\\
\rho(x) = \left|\Psi(x,t)\right|^2 &= \frac{1}{2}\left[\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_2^*\psi_1\exp(3\omega t) + \psi_1^*\psi_2\exp(-3\omega t)\right]\\
&\text{the stationary states are real, so } \psi_n^* = \psi_n\\
\rho(x) &= \frac{1}{2}\left\{\psi_1^2 + \psi_2^2 + \psi_1\psi_2\left[\exp(3\omega t) + \exp(-3\omega t)\right]\right\}\\
&\text{using Euler's formula, we can rewrite this as}\\
\rho(x) &= \frac{1}{2}\left\{\psi_1^2 + \psi_2^2 + \psi_1\psi_2\left[\cos(3\omega t) + \imath\sin(3\omega t) + \cos(-3\omega t) + \imath\sin(-3\omega t)\right]\right\}\\
&= \frac{1}{2}\left\{\psi_1^2 + \psi_2^2 + \psi_1\psi_2\left[\cos(3\omega t) + \imath\sin(3\omega t) + \cos(3\omega t) - \imath\sin(3\omega t)\right]\right\}\\
&= \frac{1}{2}\left[\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2\cos(3\omega t)\right]\\
&\text{substituting } \psi_n(x) = \sqrt{\frac{2}{\ell}}\sin\left(\frac{n\pi x}{\ell}\right)\\
\rho(x) &= \frac{1}{\ell}\left[\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) + 2\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\cos(3\omega t)\right]
\end{aligned}$$

c. Compute $\langle x\rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? The amplitude?

$$\begin{aligned}
\langle x\rangle = \langle\Psi|x\Psi\rangle = \int_0^\ell x\rho(x)\,\mathrm{d}x &= \frac{1}{\ell}\int_0^\ell x\left[\sin^2\left(\frac{\pi x}{\ell}\right) + \sin^2\left(\frac{2\pi x}{\ell}\right) + 2\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\cos(3\omega t)\right]\mathrm{d}x\\
&= \frac{1}{\ell}\left[\int_0^\ell x\sin^2\left(\frac{\pi x}{\ell}\right)\mathrm{d}x + \int_0^\ell x\sin^2\left(\frac{2\pi x}{\ell}\right)\mathrm{d}x + 2\int_0^\ell x\sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{2\pi x}{\ell}\right)\cos(3\omega t)\,\mathrm{d}x\right]\\
&= \frac{1}{\ell}\left[\frac{\ell^2}{4} + \frac{\ell^2}{4} - 2\frac{8\ell^2}{9\pi^2}\cos(3\omega t)\right] = \frac{\ell}{2} - \frac{16\ell}{9\pi^2}\cos(3\omega t)
\end{aligned}$$

The angular frequency is 3ω or $3\pi^2\hbar/2m\ell^2$, and the amplitude is $16\ell/9\pi^2$, which is approximately 0.18ℓ, well below $\ell/2$.

d. Compute $\langle p \rangle$. We could do this by finding $\langle \Psi | \hat{p} \Psi \rangle$, but as the book suggests, there is a faster way, namely, finding $\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t}$

$$\langle p \rangle = \frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\ell}{2} - \frac{16\ell}{9\pi^2}\cos(3\omega t)\right] = 0 + (3\omega)\frac{16\ell}{9\pi^2}\sin(3\omega t) = \omega\frac{16\ell}{3\pi^2}\sin(3\omega t)$$

substituting ω gives

$$\langle p \rangle = \frac{\pi^2\hbar}{2m\ell^2}\frac{16\ell}{3\pi^2}\sin(3\omega t) = \frac{8\hbar}{3m\ell}\sin(3\omega t)$$

e. If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H. How does it compare with E_1 and E_2?

The wavefunction is a superposition of only two components, $c_1\psi_1(x)\varphi_1(t)$ and $c_2\psi_2(x)\varphi_2(t)$, therefore we may rewrite it as such

$$\Psi(x,t) = \sum_{n=1}^{2} c_n\psi_n(x)\varphi_n(t),$$

where c_n has already been found to be $\sqrt{2}/2$. The probability of getting the nth energy, E_n, is given by the square of the coefficient, c_n, thus $\Pr(E_n) = |c_n|^2 = \left|\sqrt{2}/2\right|^2 = 1/2$. In other words, each of the two energies has an equal probability of being observed.

Since E_n is given by $(n\pi\hbar)^2/(2m\ell^2)$, $E_1 = \pi^2\hbar^2/2m\ell^2$, and $E_2 = 4\pi^2\hbar^2/2m\ell^2$. The expectation value of H is thus

$$\langle H \rangle = c_1E_1 + c_2E_2 = \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2m\ell^2} + \frac{4\pi^2\hbar^2}{2m\ell^2}\right) = \frac{5\pi^2\hbar^2}{4m\ell^2}$$

Problem 2.7

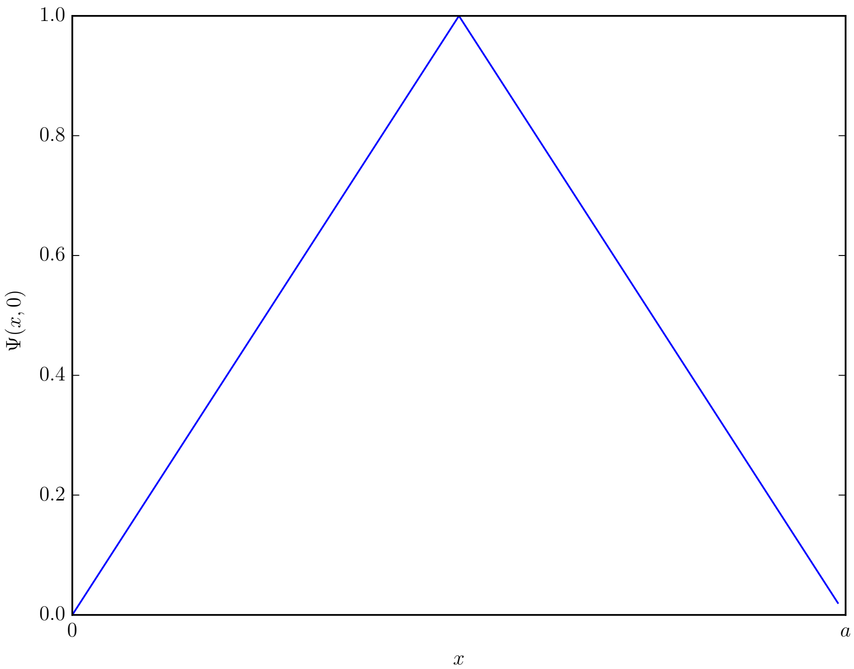
A particle in the infinite square well has the initial wave function

$$\Psi(x,0) = A\begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$

a. Sketch $\Psi(x,0)$, and determine the constant A.

$$1 = \int_{-\infty}^{\infty} \left|\Psi(x,0)\right|^2 \mathrm{d}x = |A|^2\left(\int_0^{a/2} x^2\,\mathrm{d}x + \int_{a/2}^{a}(a-x)^2\,\mathrm{d}x\right) \implies |A| = \sqrt{\frac{12}{a^3}}$$

$$\Psi(x,0) = \sqrt{\frac{12}{a^3}}\begin{cases} x, & 0 \le x \le a/2, \\ a - x, & a/2 \le x \le a. \end{cases}$$



b. Find $\Psi(x,t)$.

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

$$\implies c_n = \langle \psi_n | \Psi(x,0) \rangle = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \Psi(x,0)\,\mathrm{d}x$$

$$= \sqrt{\frac{2}{a}}\sqrt{\frac{12}{a^3}} \left[\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x + \int_{a/2}^{a} (a-x) \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x \right]$$

$$= \sqrt{\frac{24}{a^4}} \left[\frac{2a^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right] = \frac{4\sqrt{6}}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) \varphi_n(t) = \frac{4\sqrt{6}}{\pi^2}\sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\imath E_n t/\hbar\right)$$

$$= \frac{8}{\pi^2}\sqrt{\frac{3}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\imath E_n t/\hbar\right)$$

c. What is the probability that a measurement of the energy would yield the value E_1?

$$\Pr(E_1) = |c_1|^2 = \left| \frac{4\sqrt{6}}{\pi^2} \sin\left(\frac{\pi}{2}\right) \right|^2 = \frac{96}{\pi^4} \approx 0.9855$$

Interesting observation: since the nth coefficient has a factor of $\sin(n\pi/2)$, only the odd harmonics are non-zero. This is much like a clarinet, and, if our Lasso results are to be trusted, OGLE-LMC-CEP-1406.

d. Find the expectation value of the energy.

Assuming by "energy", total energy is implied, this is the expectation value of the Hamiltonian, given by Equation 2.39 as

$$\langle H\rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n = \sum_{n=1}^{\infty}\left[\underbrace{\left|\frac{4\sqrt{6}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right|^2}_{c_n}\underbrace{\frac{1}{2m}\left(\frac{n\pi\hbar}{\ell}\right)^2}_{E_n}\right]$$

pulling the terms independent of n out of the sum gives

$$\langle H\rangle = \frac{48\hbar^2}{\pi^2 m\ell^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(\frac{n\pi}{2}\right)$$

$\sin^2\left(\frac{n\pi}{2}\right)$ is zero for even n and one for odd n, allowing us to simplify the summation to

$$\langle H\rangle = \frac{48\hbar^2}{\pi^2 m\ell^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{48\hbar^2}{\pi^2 m\ell^2}\frac{\pi^2}{8} = \frac{6\hbar^2}{m\ell^2}$$

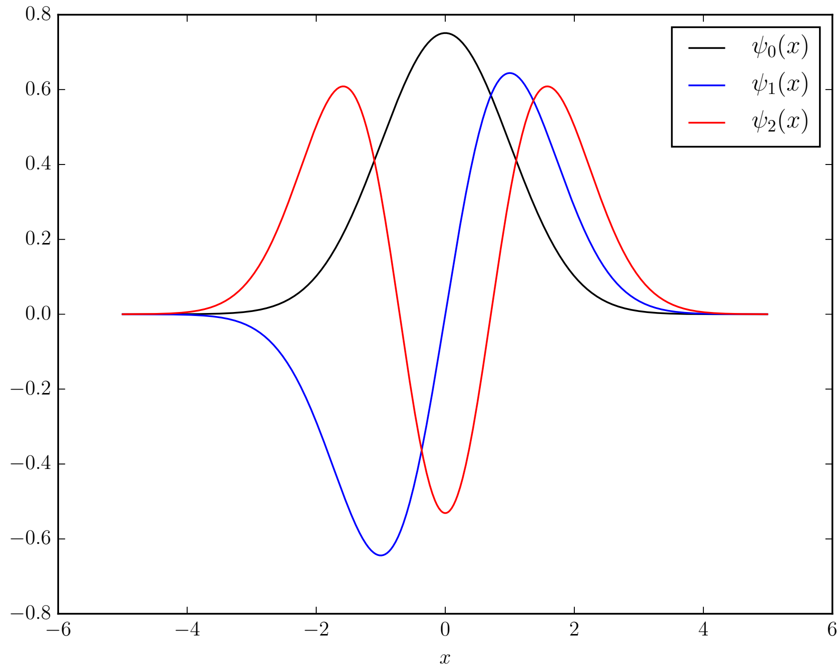
Problem 2.10

a. Construct $\psi_2(x)$

We will construct the 2nd stationary state by applying the ladder operator on $\psi_1(x)$.

$$\begin{aligned}
\psi_2(x) &= \frac{1}{\sqrt{2!}}(a_+)^2\psi_0(x) = \frac{1}{\sqrt{2!}}a_+\psi_1(x) = \\
&= \frac{1}{\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - \imath p)\sqrt{\frac{2m\omega}{\hbar}}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\hbar}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\left(m\omega x - \imath\frac{\hbar}{\imath}\frac{\partial}{\partial x}\right)x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= \frac{1}{\sqrt{2}}\frac{1}{\hbar}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\left(m\omega x - \hbar\frac{\partial}{\partial x}\right)x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= \frac{1}{\hbar\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\left\{m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right) - \hbar\frac{\partial}{\partial x}\left[x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\right\} \\
&= \frac{1}{\hbar\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\left\{m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right) - \hbar\frac{\partial x}{\partial x}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) - \hbar x\frac{\partial}{\partial x}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right\} \\
&= \frac{1}{\hbar\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\left\{m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right) - \hbar\exp\left(-\frac{m\omega}{2\hbar}x^2\right) - \hbar x\left(-\frac{m\omega}{2\hbar}2x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right\} \\
&= \frac{1}{\hbar\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left\{m\omega x^2 - \hbar - \hbar x\left(-\frac{m\omega}{2\hbar}2x\right)\right\} \\
&= \frac{1}{\hbar\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(m\omega x^2 - \hbar + m\omega x^2\right) = \frac{1}{\hbar\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(2m\omega x^2 - \hbar\right) \\
&= \frac{1}{\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(\frac{2m\omega}{\hbar}x^2 - 1\right)
\end{aligned}$$

b. Sketch ψ_0, ψ_1, and ψ_2.



c. Check the orthogonality of ψ_0, ψ_1, and ψ_2, by explicit integration. *Hint:* If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.

As the book suggests, we may exploit the fact that the even-numbered stationary states are even, while the odd-numbered ones are oddl, as can be seen from the plot above. This means ψ_0 and ψ_2 are both orthogonal to ψ_1, and we merely need to check the orthogonality of ψ_0 and ψ_2. If they are orthogonal, their inner product should be zero.

$$
\begin{aligned}
\langle\psi_0|\psi_2\rangle &= \int_{-\infty}^{\infty} \psi_0^*(x)\psi_2(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi_0(x)\psi_2(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\frac{1}{\sqrt{2}}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(\frac{2m\omega}{\hbar}x^2-1\right)\mathrm{d}x \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{m\omega}{\hbar}x^2\right)\left(\frac{2m\omega}{\hbar}x^2-1\right)\mathrm{d}x \\
&\text{Let } u = m\omega/\hbar \\
\langle\psi_0|\psi_2\rangle &= \frac{1}{\sqrt{2}}\sqrt{\frac{u}{\pi}}\int_{-\infty}^{\infty}\exp\left(-ux^2\right)\left(2ux^2-1\right)\mathrm{d}x = 0
\end{aligned}
$$