Griffiths Chapter 3

Dan Wysocki

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Problem 3.1

a. Show that the set of all square-integrable functions is a vector space (refer to Section A.1 for the definition). *Hint:* The main problem is to show that the sum of two square integrable functions is itself square integrable. Use Equation 3.7. Is the set of all *normalized* functions a vector space?

A vector space is a set of vectors which are closed under vector addition, and scalar multiplication. More precisely, a set S is a vector space if the following two properties hold true.

$$
\begin{aligned}
\left|\alpha\right\rangle, \left|\beta\right\rangle \in S &\implies \left|\alpha\right\rangle + \left|\beta\right\rangle \in S \\
\left|\alpha\right\rangle \in S \land c \in \mathbb{R} &\implies c\left|\alpha\right\rangle \in S
\end{aligned}
$$

We want to show that $\left|\alpha\right\rangle, \left|\beta\right\rangle \in L_2(a,b) \implies \left|\alpha\right\rangle + \left|\beta\right\rangle \in L_2(a,b)$.

By the definition of $L_2(a,b)$, we know that

$$\left|\alpha\right\rangle \in L_2(a,b) \implies \langle\alpha|\alpha\rangle < \infty \text{ and } \left|\beta\right\rangle \in L_2(a,b) \implies \langle\beta|\beta\rangle < \infty \tag{1}$$

By the Schwarz inequality (Equation 3.7), we also know that

$$\langle\alpha|\alpha\rangle < \infty \text{ and } \langle\beta|\beta\rangle < \infty \implies \langle\alpha|\beta\rangle < \infty \text{ and } \langle\beta|\alpha\rangle < \infty \tag{2}$$

Given these two things, let us now test if $\left|\alpha\right\rangle + \left|\beta\right\rangle \in L_2(a,b)$.

$$
\begin{aligned}
\langle\alpha+\beta|\alpha+\beta\rangle &= \int_a^b (\alpha+\beta)^*(\alpha+\beta)\,\mathrm{d}x = \int_a^b (\alpha^*\alpha + \beta^*\beta + \alpha^*\beta + \beta^*\alpha)\,\mathrm{d}x \\
&= \langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle + \langle\alpha|\beta\rangle + \langle\beta|\alpha\rangle
\end{aligned} \tag{3}
$$

From (1) and (2), we know that each term in (3) is a finite complex number, and since complex numbers are closed under addition, the sum must also be a finite complex number. Therefore $\langle\alpha+\beta|\alpha+\beta\rangle < \infty$, and likewise $\left|\alpha\right\rangle + \left|\beta\right\rangle \in L_2(a,b)$.

Now we must prove that $L_2(a,b)$ is closed under scalar multiplication.

Let $\left|\alpha\right\rangle \in L_2(a,b)$, and $c \in \mathbb{R}$. We want to show that $c\left|\alpha\right\rangle \in L_2(a,b)$. $\langle c\alpha|c\alpha\rangle = c^2\langle\alpha|\alpha\rangle$. Since both c and $\left|\alpha\right\rangle$ are finite complex numbers, their product must be as well, meaning $c^2\langle\alpha|\alpha\rangle < \infty$, and therefore $c\left|\alpha\right\rangle \in L_2(a,b)$.

We have proven that $L_2(a,b)$ is closed under vector addition and scalar multiplication, and therefore $L_2(a,b)$ is a vector space.

On the contrary, the set of all *normalized* functions is *not* a vector space. While it may be a subset of $L_2(a,b)$, it is not itself closed under vector addition and scalar multiplication. Consider a vector $\left|\gamma\right\rangle$, such that $\langle\gamma|\gamma\rangle = 1$, as well as an arbitrary scalar c. $\langle c\gamma|c\gamma\rangle = |c|^2$, and since c is not strictly ± 1, $\langle c\gamma|c\gamma\rangle \neq 1$, and thus the set of normalized functions is *not* a vector space.

b. Show that the integral in Equation 3.6 satisfies the conditions for an inner product (Section A.2).

There are four conditions that Equation 3.6 must meet in order to satisfy the conditions of inner products. The first is $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$. By Equation 3.6,

$$\langle\beta|\alpha\rangle = \int_a^b \beta^*\alpha\,\mathrm{d}x = \left(\int_a^b \beta\alpha^*\,\mathrm{d}x\right)^* = \left(\int_a^b \alpha^*\beta\,\mathrm{d}x\right)^* = \langle\alpha|\beta\rangle^*$$

The second is $\langle\alpha|\alpha\rangle \geq 0$.

$$\langle\alpha|\alpha\rangle = \int_a^b \alpha^*\alpha\,\mathrm{d}x = \int_a^b |\alpha|^2\,\mathrm{d}x$$

Since $|\alpha|^2$ is strictly real and non-negative, its integral must also be, meaning Equation 3.6 satisfies the property $\langle\alpha|\alpha\rangle \geq 0$.

The third is closely related to the second: $\langle\alpha|\alpha\rangle = 0 \iff |\alpha\rangle = |0\rangle$. Since this is a two-way implication, we must prove each way separately. We prove the forward implication by contrapositive: $|\alpha\rangle \neq |0\rangle \implies \langle\alpha|\alpha\rangle = \int_a^b |\alpha|^2\,\mathrm{d}x > 0 \implies \langle\alpha|\alpha\rangle \neq 0$. The reverse implication is proven directly: $|\alpha\rangle = |0\rangle \implies \langle\alpha|\alpha\rangle = \int_a^b |0|^2\,\mathrm{d}x = 0\int_a^b \mathrm{d}x = 0$.

The fourth condition is the distributive property: $\langle\alpha|\left(b\,|\beta\rangle + c\,|\gamma\rangle\right) = b\,\langle\alpha|\beta\rangle + c\,\langle\alpha|\gamma\rangle$

$$\begin{aligned}
\langle\alpha|\left(b\,|\beta\rangle + c\,|\gamma\rangle\right) &= \int_\mu^\nu \alpha^*(b\beta + c\gamma)\,\mathrm{d}x \\
&= \int_\mu^\nu (\alpha^* b\beta + \alpha^* c\gamma)\,\mathrm{d}x \\
&= \int_\mu^\nu \alpha^* b\beta\,\mathrm{d}x + \int_\mu^\nu \alpha^* c\gamma\,\mathrm{d}x \\
&= b\int_\mu^\nu \alpha^*\beta\,\mathrm{d}x + c\int_\mu^\nu \alpha^*\gamma\,\mathrm{d}x \\
&= b\,\langle\alpha|\beta\rangle + c\,\langle\alpha|\gamma\rangle
\end{aligned}$$

And thus we have shown that the integral satisfies the conditions for an inner product.

Problem 3.4

a. Show that the *sum* of two hermitian operators is hermitian.

Let $\hat{Q}_1$ and $\hat{Q}_2$ be hermitian operators. By Equation 3.16, this means that $\langle f|\hat{Q}_1 f\rangle = \langle\hat{Q}_1 f|f\rangle$ and $\langle f|\hat{Q}_2 f\rangle = \langle\hat{Q}_2 f|f\rangle$. We want to show that $(\hat{Q}_1 + \hat{Q}_2)$ is also a hermitian operator, i.e. $\langle f|(\hat{Q}_1 + \hat{Q}_2)f\rangle = \langle(\hat{Q}_1 + \hat{Q}_2)f|f\rangle$.

$$\begin{aligned}
\langle f|(\hat{Q}_1 + \hat{Q}_2)f\rangle &= \langle f|\hat{Q}_1 f\rangle + \langle f|\hat{Q}_2 f\rangle && \text{(by distributive property)} \\
&= \langle\hat{Q}_1 f|f\rangle + \langle\hat{Q}_2 f|f\rangle && \text{(by Equation 3.16)} \\
&= \langle(\hat{Q}_1 + \hat{Q}_2)f|f\rangle
\end{aligned}$$

Therefore, $(\hat{Q}_1 + \hat{Q}_2)$ is a hermitian operator.

b. Suppose $\hat{Q}$ is hermitian, and α is a complex number. Under what condition (on α) is $\alpha\hat{Q}$ hermitian?

There are several cases to consider. Note that in each case, since $\hat{Q}$ is hermitian, $(\alpha\hat{Q})^* = \alpha^*\hat{Q}$.

If $\mathrm{Re}(\alpha) = 0$, and $\mathrm{Im}(\alpha) \neq 0$, then $\alpha^* = -\alpha$, and therefore $(\alpha\hat{Q})^* = -\alpha\hat{Q} \neq \alpha\hat{Q}$.

If $\mathrm{Re}(\alpha) \neq 0$, and $\mathrm{Im}(\alpha) \neq 0$, then α^* is a complex number which is neither α or $-\alpha$, and $(\alpha\hat{Q})^* \neq \alpha\hat{Q}$.

Finally, if $\alpha \in \mathbb{R}$, then $\alpha^* = \alpha$, and therefore $(\alpha\hat{Q})^* = \alpha\hat{Q}$. So if $\hat{Q}$ is hermitian, and α is complex, then $\alpha\hat{Q}$ is hermitian iff α is real.

c. When is the *product* of two hermitian operators hermitian?

Let $\hat{Q}_1$ and $\hat{Q}_2$ be hermitian operators. We want to show that $\langle f|\hat{Q}_1\hat{Q}_2 f\rangle = \langle \hat{Q}_1\hat{Q}_2 f|f\rangle$. We start with the left hand side:

$$\begin{aligned}
\langle f|\hat{Q}_1\hat{Q}_2 f\rangle &= \langle f|\hat{Q}_1(\hat{Q}_2 f)\rangle && \text{(by associativity of operators)} \\
&= \langle \hat{Q}_1 f|\hat{Q}_2 f\rangle && \text{(because } \hat{Q}_1 \text{ is hermitian)} \\
&= \langle \hat{Q}_2(\hat{Q}_1 f)|f\rangle && \text{(because } \hat{Q}_2 \text{ is hermitian)}
\end{aligned}$$

Now, as stated earlier, $\hat{Q}_1\hat{Q}_2$ is hermitian iff $\langle f|\hat{Q}_1\hat{Q}_2 f\rangle = \langle \hat{Q}_1\hat{Q}_2 f|f\rangle$, and since we have just shown that $\langle f|\hat{Q}_1\hat{Q}_2 f\rangle = \langle \hat{Q}_2\hat{Q}_1 f|f\rangle$, we are left with the condition that $\hat{Q}_1\hat{Q}_2$ must be equal to $\hat{Q}_2\hat{Q}_1$, which is only true if the two operators commute. Therefore, $\hat{Q}_1\hat{Q}_2$ is hermitian iff $[\hat{Q}_1, \hat{Q}_2] = 0$.

d. Show that the position operator and the hamiltonian operator are hermitian.

Showing that the position operator is hermitian is trivial, because $\hat{x} = x$, which is a multiplication operator, and therefore holds the commutative property of multiplication, meaning

$$\langle f|xf\rangle = \int_a^b f^* x f \,\mathrm{d}x = \int_a^b x f^* f = \langle xf|f\rangle\,.$$

Before showing that $\hat{H}$ is hermitian, it is helpful to point out that each of its components is hermitian. $-(\hbar^2/2m)$ is a real number, and therefore hermitian. $V(x)$ is a real-valued function, and is therefore hermitian as well. We will show in Problem 3.5.a that the adjoint $\frac{\mathrm{d}}{\mathrm{d}x}$ is $-\frac{\mathrm{d}}{\mathrm{d}x}$, and so it follows that the adjoint of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is $(-1)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ (i.e. it is hermitian). Thus, we can simply use some simple bra–ket manipulations to prove that $\hat{H}$ is hermitian.

We want to show that $\langle f|\hat{H}g\rangle = \langle \hat{H}f|g\rangle$.

$$\begin{aligned}
\langle f|\hat{H}g\rangle &= \left\langle f\middle|\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)g\right\rangle = \left\langle f\middle|\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)g\right\rangle + \langle f\big|V(x)g\rangle \\
&= \left\langle\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)f\middle|g\right\rangle + \langle V(x)f\big|g\rangle = \left\langle\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)f\middle|g\right\rangle = \langle\hat{H}f|g\rangle
\end{aligned}$$

Problem 3.5 The *hermitian conjugate* (or *adjoint*) of an operator $\hat{Q}$ is the operator $\hat{Q}^\dagger$ such that

$$\langle f|\hat{Q}g\rangle = \langle\hat{Q}^\dagger f|g\rangle\,\forall f, g.$$

(A hermitian operator, then, is equal to its hermitian conjugate: $\hat{Q} = \hat{Q}^\dagger$.)

a. Find the hermitian conjugates of x, $\imath$, and $\frac{\mathrm{d}}{\mathrm{d}x}$.

x is trivial, since it is a hermitian operator, but we will do the explicit proof for the sake of completeness.

$$
\begin{aligned}
\langle f|xg\rangle &= \int_{-\infty}^{\infty} f(x)^* x g(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x f(x)^* g(x)\,\mathrm{d}x && \text{(commutative property of multiplication)} \\
&= \int_{-\infty}^{\infty} (x f(x))^* g(x)\,\mathrm{d}x && (x \in \mathbb{R} \implies x = x^*) \\
&= \langle xf|g\rangle \implies x^\dagger = x \\
\langle f|\imath g\rangle &= \int_{-\infty}^{\infty} f(x)^* \imath g(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \imath f(x)^* g(x)\,\mathrm{d}x && \text{(commutative property of multiplication)} \\
&= \int_{-\infty}^{\infty} (-\imath f(x))^* g(x)\,\mathrm{d}x && (\imath^* = -\imath) \\
&= \langle (-\imath) f|g\rangle \implies -\imath = \imath^\dagger = \imath^* \\
\langle f|\tfrac{\mathrm{d}}{\mathrm{d}x} g\rangle &= \int_{-\infty}^{\infty} f(x)^* \frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x \\
\text{Let } u = f(x)^* &\implies \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x)^* \implies \mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}x} f(x)^*\,\mathrm{d}x \\
\text{Let } \mathrm{d}v = \frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x &\implies \int \mathrm{d}v = \int \frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x \implies v = g(x) \\
\langle f|\tfrac{\mathrm{d}}{\mathrm{d}x} g\rangle &= \left[f(x)^* g(x) - \int g(x) \frac{\mathrm{d}}{\mathrm{d}x} f(x)^*\,\mathrm{d}x \right]_{-\infty}^{\infty}
\end{aligned}
$$

Using the fact that functions in Hilbert space must go to zero at $\pm\infty$, we can zero out $f(x)^* g(x)$, leaving us with $\langle f|\frac{\mathrm{d}}{\mathrm{d}x} g\rangle = \int_{-\infty}^{\infty} \left(-\frac{\mathrm{d}}{\mathrm{d}x} f(x)^*\right) g(x)\,\mathrm{d}x$, meaning $\frac{\mathrm{d}}{\mathrm{d}x}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x}$.

b. Construct the hermitian conjugate of the harmonic oscillator raising operator, a_+ (Equation 2.47).

This will be trivial if we first prove the distributive property of adjoints, which states that for operators $\hat{Q}$ and $\hat{R}$, $(\hat{Q} + \hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger$. So we want to show that $\langle f|(\hat{Q} + \hat{R})g\rangle = \langle (\hat{Q}^\dagger + \hat{R}^\dagger) f|g\rangle$.

$$
\begin{aligned}
\langle f|(\hat{Q} + \hat{R})g\rangle &= \langle f|\hat{Q}g\rangle + \langle f|\hat{R}g\rangle \\
&= \langle \hat{Q}^\dagger f|g\rangle + \langle \hat{R}^\dagger f|g\rangle \\
&= \langle (\hat{Q} + \hat{R})^\dagger f|g\rangle \implies (\hat{Q} + \hat{R})^\dagger = \hat{Q}^\dagger + \hat{R}^\dagger
\end{aligned}
$$

Now, using this property, along with the property we are about to show in part c, and the fact that x and p are hermitian, the solution becomes straightforward:

$$
\begin{aligned}
a_+^\dagger &= \left[\frac{1}{\sqrt{2\hbar m\omega}} (m\omega x - \imath p) \right]^\dagger = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega x - \imath p)^\dagger \\
&= \frac{1}{\sqrt{2\hbar m\omega}} ((m\omega x)^\dagger - (\imath p)^\dagger) = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega x + \imath p) = a_-
\end{aligned}
$$

c. Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$.

$$\langle f|(\hat{Q}\hat{R})g\rangle = \langle \hat{Q}^\dagger f|\hat{R}g\rangle = \langle \hat{R}^\dagger\hat{Q}^\dagger f|g\rangle = \langle(\hat{Q}\hat{R})^\dagger f|g\rangle \implies (\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$$

Problem 3.6 Consider the operator $\hat{Q} = \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}$, where (as in Example 3.1) ϕ is the azimuthal angle in polar coordinates, and the functions are subject to Equation 3.26. Is $\hat{Q}$ hermitian? Find its eigenfunctions and eigenvalues. What is the spectrum of $\hat{Q}$? Is the spectrum degenerate?

To see if $\hat{Q}$ is hermitian, we expand $\langle f|\hat{Q}g\rangle$.

$$\begin{aligned}
\langle f|\hat{Q}g\rangle &= \int_0^{2\pi} f(\phi)^* \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} g(\phi)\,\mathrm{d}\phi \\
\text{Let } u = f(\phi)^* &\implies \frac{\mathrm{d}u}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi} f(\phi)^* \implies \mathrm{d}u = \frac{\mathrm{d}}{\mathrm{d}\phi} f(\phi^*)\,\mathrm{d}\phi \\
\text{Let } \mathrm{d}v = \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} g(\phi)\,\mathrm{d}\phi &\implies \int \mathrm{d}v = \int \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} g(\phi)\,\mathrm{d}\phi \implies v = \frac{\mathrm{d}g}{\mathrm{d}\phi} \\
\langle f|\hat{Q}g\rangle &= \left[f(\phi)^* \frac{\mathrm{d}g}{\mathrm{d}\phi} - \int \frac{\mathrm{d}g}{\mathrm{d}\phi}\frac{\mathrm{d}f^*}{\mathrm{d}\phi}\,\mathrm{d}\phi \right]_0^{2\pi}
\end{aligned}$$

Using the fact that $f(\phi) = f(\phi + 2\pi)$, we can zero out the term outside the integral.

$$\begin{aligned}
\langle f|\hat{Q}g\rangle &= -\int_0^{2\pi} \frac{\mathrm{d}g}{\mathrm{d}\phi}\frac{\mathrm{d}f^*}{\mathrm{d}\phi}\,\mathrm{d}\phi \\
\text{Let } u = \frac{\mathrm{d}f^*}{\mathrm{d}\phi} &\implies \mathrm{d}u = \frac{\mathrm{d}^2 f^*}{\mathrm{d}\phi^2}\,\mathrm{d}\phi \\
\text{Let } \mathrm{d}v = \mathrm{d}g\phi\,\mathrm{d}\phi &\implies v = g(\phi) \\
\langle f|\hat{Q}g\rangle &= -\left[\frac{\mathrm{d}f^*}{\mathrm{d}\phi} g(\phi) - \int g(\phi)\frac{\mathrm{d}^2 f^*}{\mathrm{d}\phi^2}\,\mathrm{d}\phi \right]_0^{2\pi} \\
&= +\int g(\phi)\frac{\mathrm{d}^2 f^*}{\mathrm{d}\phi^2}\,\mathrm{d}\phi = \langle \tfrac{\mathrm{d}^2}{\mathrm{d}\phi^2} f|g\rangle \implies \hat{Q}^\dagger = \tfrac{\mathrm{d}^2}{\mathrm{d}\phi^2}
\end{aligned}$$

Problem 3.7

a. Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\hat{Q}$, with the same eigenvalue q. Show that any linear combination of f and g is itself an eigenfunction of $\hat{Q}$, with eigenvalue q.

We know that $\hat{Q}f = qf$ and $\hat{Q}g = qg$. We multiply each of the equations by their own scalars, which we call a and b, giving $\hat{Q}(af) = q(af)$ and $\hat{Q}(bg) = q(bg)$. Adding the two equations together gives

$$\begin{aligned}
\hat{Q}(af) + \hat{Q}(bg) &= q(af) + q(bg) \\
\hat{Q}(af + bg) &= q(af + bg)
\end{aligned}$$

Which means that $af + bg$ is an eigenfunction of $\hat{Q}$ with eigenvalue q, and since a and b are arbitrary scalars, this holds true for any linear combination of f and g.

b. Check that $f(x) = \exp(x)$ and $g(x) = \exp(-x)$ are eigenfunctions of the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, with the same eigenvalue. Construct two linear combinations of f and g that are *orthogonal* eigenfunctions on the interval $(-1, 1)$.

First, let us note that $\exp(cx)$ is an eigenfunction of $\frac{\mathrm{d}}{\mathrm{d}x}$, with eigenvalue c, which is a property every Calculus 1 student encounters without realizing it. It follows that $\exp(cx)$ is an eigenfunction of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, with eigenvalue c^2. Therefore: $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = (1)^2 f(x)$, and $\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x) = (-1)^2 g(x)$, meaning that both $f(x)$ and $g(x)$ are eigenfunctions of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, with eigenvalue 1.

Two linear combinations of f and g are orthogonal if their inner product is zero. We construct two arbitrary linear combinations: $u(x) = af(x) + bg(x)$ and $v(x) = cf(x) + dg(x)$, which we know are also eigenfunctions of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with eigenvalue 1, from part a. Now we take their inner product on the interval $(-1, 1)$, set it equal to zero, and find coefficients a, b, c, and d which satisfy the condition.

$$\begin{aligned}
\langle u|v\rangle &= \int_{-1}^{1} (a\exp(x) + b\exp(-x))^*(c\exp(x) + d\exp(-x))\,\mathrm{d}x \\
&= \int_{-1}^{1} (ac\exp(2x) + ad\exp(0) + bc\exp(0) + bd\exp(-2x))\,\mathrm{d}x \\
&= ac\int_{-1}^{1}\exp(2x)\,\mathrm{d}x + ad\int_{-1}^{1}\mathrm{d}x + bc\int_{-1}^{1}\mathrm{d}x + bd\int_{-1}^{1}\exp(-2x)\,\mathrm{d}x \\
&= ac\sinh(2) + ad(1+1) + bc(1+1) + bd\sinh(2) \\
&= \sinh(2)(ac+bd) + 2(ad+bc) := 0 \\
&\implies 2(ad+bc) = -\sinh(2)(ac+bd) \\
&\implies \frac{ad+bc}{ac+bd} = -\frac{\sinh(2)}{2}
\end{aligned}$$

This has infinitely many solutions. To get a particular solution, we solve for the constant a in terms of b, c and d, and then choose arbitrary values for b, c, and d, for which a is well defined.

$$\begin{aligned}
\frac{ad+bc}{ac+bd} &= -\frac{\sinh(2)}{2} \\
2ad + 2bc &= -ac\sinh(2) - bd\sinh(2) \\
2ad + ac\sinh(2) &= -2bc - bd\sinh(2) \\
a(2d + c\sinh(2)) &= -2bc - bd\sinh(2) \\
a &= -\frac{2bc + bd\sinh(2)}{2d + c\sinh(2)}
\end{aligned}$$

The only constraint here is that $2d + c\sinh(2) \neq 0$, so I will choose positive integer values for b, c, and d, which will ensure nothing cancels with the transcendental number $\sinh(2)$. Let $b = 2$, $c = 3$, and $d = 4$. This means that a is given by

$$a = -\frac{2\cdot 2\cdot 3 + 2\cdot 4\sinh(2)}{2\cdot 4 + 3\sinh(2)} = -\frac{12 + 8\sinh(2)}{8 + 3\sinh(2)} \approx -2.172$$

Our particular linear combination eigenfunctions $u(x)$ and $v(x)$ are thus given by

$$u(x) = -\frac{12 + 8\sinh(2)}{8 + 3\sinh(2)}\exp(x) + 2\exp(-x), \text{ and } v(x) = 3\exp(x) + 4\exp(-x).$$

Problem 3.13

a. Prove the following commutator identity:

$$[AB, C] = A[B, C] + [A, C]B$$

Recall $[X, Y] = XY - YX$, it follows that

$$\begin{align} [AB, C] &= ABC - CBA, \tag{1} \\ [B, C] &= BC - CB \implies BC = [B, C] + CB, \tag{2} \\ [A, C] &= AC - CA \implies -CA = [A, C] - AC, \tag{3} \end{align}$$

Substituting (2) and (3) into (1) gives:

$$\begin{aligned} [AB, C] &= A([B, C] + CB) + ([A, C] - AC)B \\ &= A[B, C] + ACB + [A, C]B - ACB \\ &= A[B, C] + [A, C]B \end{aligned}$$

b. Show that

$$[x^n, p] = \imath\hbar n x^{n-1}$$

To avoid confusion, we first find $[x^n, p]f$, where $f(x)$ is an arbitrary test function.

$$\begin{aligned} [x^n, p] = x^n p(f) - p x^n(f) &= x^n \frac{\hbar}{\imath} \frac{\partial}{\partial x}(f) - \frac{\hbar}{\imath} \frac{\partial}{\partial x}(x^n f) \\ &= \frac{\hbar}{\imath} \left[x^n \frac{\partial f}{\partial x} - \left(x^n \frac{\partial f}{\partial x} + \frac{\partial x^n}{\partial x}(f) \right) \right] = -\frac{\hbar}{\imath} \frac{\partial}{\partial x} x^n(f) \\ &= \imath\hbar n x^{n-1}(f) \end{aligned}$$

Removing the test function gives $[x^n, p] = \imath\hbar n x^{n-1}$.

c. Show more generally that

$$[f(x), p] = \imath\hbar \frac{\mathrm{d}f}{\mathrm{d}x},$$

for any function $f(x)$.

Like before, we begin by inserting an arbitrary test function, $g(x)$, finding $[f(x), p]g(x)$:

$$\begin{aligned} [f(x), p]g(x) &= f(x) \frac{\hbar}{\imath} \frac{\partial}{\partial x} g(x) - \frac{\hbar}{\imath} \frac{\partial}{\partial x} [f(x) g(x)] \\ &= \frac{\hbar}{\imath} \left\{ f(x) \frac{\mathrm{d}g}{\mathrm{d}x} - \left[f(x) \frac{\mathrm{d}g}{\mathrm{d}x} + g(x) \frac{\mathrm{d}f}{\mathrm{d}x} \right] \right\} \\ &= \frac{\hbar}{\imath} \left[f(x) \frac{\mathrm{d}g}{\mathrm{d}x} - f(x) \frac{\mathrm{d}g}{\mathrm{d}x} - g(x) \frac{\mathrm{d}f}{\mathrm{d}x} \right] \\ &= -\frac{\hbar}{\imath} \frac{\mathrm{d}f}{\mathrm{d}x} g(x) = \imath\hbar \frac{\mathrm{d}f}{\mathrm{d}x} g(x) \end{aligned}$$

Removing the test function gives $[f(x), p] = \imath\hbar \frac{\mathrm{d}f}{\mathrm{d}x}$.