Griffiths Chapter 3 (2)

Dan Wysocki

March 5, 2015

Problem 3.16 Solve $(p - \langle p \rangle)\Psi = \imath a(x - \langle x \rangle)\Psi$ for $\Psi(x)$. Note that $\langle x \rangle$, $\langle p \rangle$, and a are real constants.

$$\begin{aligned}
p\Psi &= \left[\imath a(x - \langle x \rangle) + \langle p \rangle\right]\Psi \\
\frac{\mathrm{d}\Psi}{\mathrm{d}x} &= \frac{\imath}{\hbar}\left[\imath a(x - \langle x \rangle) + \langle p \rangle\right]\Psi \\
\frac{\mathrm{d}\Psi}{\Psi} &= \left[-\frac{a}{\hbar}(x - \langle x \rangle) + \frac{\imath \langle p \rangle}{\hbar}\right]\mathrm{d}x \\
\int \frac{1}{\Psi}\,\mathrm{d}\Psi &= \int \left[-\frac{a}{\hbar}(x - \langle x \rangle) + \frac{\imath \langle p \rangle}{\hbar}\right]\mathrm{d}x \\
\ln\Psi &= -\frac{a}{2\hbar}(x - \langle x \rangle)^2 + \frac{\imath \langle p \rangle x}{\hbar} + C \\
\Psi(x) &= \exp(C)\exp\left(-\frac{a}{2\hbar}(x - \langle x \rangle)^2\right)\exp\left(\frac{\imath \langle p \rangle x}{\hbar}\right) \\
&= A\exp\left(-\frac{a}{2\hbar}(x - \langle x \rangle)^2\right)\exp\left(\frac{\imath \langle p \rangle x}{\hbar}\right)
\end{aligned}$$

Problem 3.18 Test the energy-time uncertainty principle for the wave function in Problem 2.5 and the observable x, by calculating σ_H, σ_x, and $\mathrm{d}\langle x \rangle / \mathrm{d}t$ exactly.

The wave function in Problem 2.5 is given by

$$\Psi(x,t) = \frac{\sqrt{2}}{2}[\psi_1(x)\varphi_1(t) + \psi_2(x)\varphi_2(t)], \text{ where } \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), \text{ and } \varphi_n(t) = \exp\left(-n^2\omega t\right)$$

or more simply

$$\Psi(x,t) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi x}{a}\right)\exp(-\omega t) + \sin\left(\frac{2\pi x}{a}\right)\exp(-4\omega t)\right],$$

and solving for the probability density function we get

$$\rho(x) = \left|\Psi(x,t)\right|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right].$$

Note that all of the above was done in the 2nd homework assignment. For more detailed steps please refer to that.

Since σ_Q is given by $\sqrt{\langle Q^2 \rangle - \langle Q \rangle^2}$ for any operator Q, we will first have to find some expectation values.

$\langle x \rangle$ was shown in the 2nd assignment to be

$$\langle x \rangle = \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t),$$

and $\mathrm{d}\langle x \rangle / \mathrm{d}t$ was shown to be

$$\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}t} = \frac{8\hbar}{3ma}\sin(3\omega t).$$

Now we will find $\left\langle x^2 \right\rangle$

$$\langle x^2\rangle = \langle\Psi|x^2\Psi\rangle = \int_0^a x^2\rho(x)\,\mathrm{d}x = \frac{1}{a}\int_0^a x^2\left[\sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\right]$$

$$= \frac{1}{a}\left[\int_0^a x^2\sin^2\left(\frac{\pi x}{a}\right)\mathrm{d}x + \int_0^a x^2\sin^2\left(\frac{2\pi x}{a}\right)\mathrm{d}x + 2\cos(3\omega t)\int_0^a x^2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\mathrm{d}x\right]$$

(integration done with SymPy and Wolfram|Alpha)

$$\langle x^2\rangle = \frac{1}{a}\left\{\left[\frac{a^3}{6} - \frac{a^3}{4\pi^2}\right] + \left[\frac{a^3}{6} - \frac{a^3}{16\pi^2}\right] + 2\cos(3\omega t)\left[-\frac{8a^3}{9\pi^2}\right]\right\}$$

$$= \frac{a^3}{a}\left[\frac{1}{6} - \frac{1}{4\pi^2} + \frac{1}{6} - \frac{1}{16\pi^2} - 2\cos(3\omega t)\frac{8}{9\pi^2}\right] = a^2\left[\frac{1}{3} - \frac{5}{16\pi^2} - 2\cos(3\omega t)\frac{8}{9\pi^2}\right]$$

(note that $\mathrm{lcm}(3, 16\pi^2, 9\pi^2) = 144\pi^2$)

$$\langle x^2\rangle = \frac{a^2}{144\pi^2}\left[48\pi^2 - 5\cdot 9 - 2\cos(3\omega t)16\cdot 8\right] = \frac{a^2}{144\pi^2}\left[48\pi^2 - 45 - 256\cos(3\omega t)\right]$$

$$= \left(\frac{a}{12\pi}\right)^2\left[48\pi^2 - 45 - 256\cos(3\omega t)\right]$$

$$\langle x\rangle^2 = a^2\left[\frac{1}{2} - \frac{16}{9\pi^2}\cos(3\omega t)\right]^2 = a^2\left[\frac{1}{4} - \frac{16}{9\pi^2}\cos(3\omega t) + \left(\frac{16}{9\pi^2}\cos(3\omega t)\right)^2\right]$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left\{\frac{48}{144} - \frac{45}{144\pi^2} - \frac{16}{9\pi^2}\cos(3\omega t) - \frac{1}{4} + \frac{16}{9\pi^2}\cos(3\omega t) + \left(\frac{16}{9\pi^2}\cos(3\omega t)\right)^2\right\}$$

$$= a^2\left[\frac{1}{12} - \frac{5}{16\pi^2} + \left(\frac{16}{9\pi^2}\cos(3\omega t)\right)^2\right]$$

$\langle H\rangle$ was also found in the previous assignment, and has value

$$\langle H\rangle = \frac{5\pi^2\hbar^2}{4ma^2}.$$

Now, instead of finding $\langle H^2\rangle$, we can invoke Equation 3.21 to skip directly to σ_H.

$$\sigma_H^2 = \langle (\hat{H} - \langle H \rangle)\Psi | (\hat{H} - \langle H \rangle)\Psi \rangle \text{, where}$$

$$(\hat{H} - \langle H \rangle) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \langle H \rangle$$

$$\sigma_H^2 = \int_0^a \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} - \langle H \rangle \Psi^* \right) \left(-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \langle H \rangle \Psi \right) \mathrm{d}x$$

$$= \int_0^a \left[\frac{\hbar^4}{4m^2}\frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial^2 \Psi}{\partial x^2} + \langle H \rangle^2 \Psi^* \Psi + \langle H \rangle \Psi \frac{\hbar^2}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \langle H \rangle \Psi^* \frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} \right] \mathrm{d}x$$

$$= \frac{\hbar^4}{4m^2}\int_0^a \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial^2 \Psi}{\partial x^2}\,\mathrm{d}x + \langle H \rangle^2 \int_0^a \Psi^* \Psi \,\mathrm{d}x + \langle H \rangle \frac{\hbar^2}{2m}\int_0^a \Psi \frac{\partial^2 \Psi^*}{\partial x^2}\,\mathrm{d}x + \langle H \rangle \frac{\hbar^2}{2m}\int_0^a \Psi^* \frac{\partial^2 \Psi}{\partial x^2}\,\mathrm{d}x$$

note that Ψ is real-valued, and therefore the complex conjugate is the identity function

$$\sigma_H^2 = \frac{\hbar^4}{4m^2}\int_0^a \left(\frac{\partial^2 \Psi}{\partial x^2}\right)^2 \mathrm{d}x + \langle H \rangle^2 \int_0^a \Psi(x)^2 \,\mathrm{d}x + \langle H \rangle \frac{\hbar^2}{2m}\int_0^a \Psi \frac{\partial^2 \Psi}{\partial x^2}\,\mathrm{d}x + \langle H \rangle \frac{\hbar^2}{2m}\int_0^a \Psi \frac{\partial^2 \Psi}{\partial x^2}\,\mathrm{d}x$$

$$= \frac{\hbar^4}{4m^2}\int_0^a \left(\frac{\partial^2 \Psi}{\partial x^2}\right)^2 \mathrm{d}x + \langle H \rangle^2 + 2\langle H \rangle \frac{\hbar^2}{2m}\int_0^a \Psi \frac{\partial^2 \Psi}{\partial x^2}\,\mathrm{d}x$$

using a computer algebra system I find the derivatives of Ψ, and evaluate the integrals

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{\pi^2}{\sqrt{a^5}}\left(\exp(3\omega t) + 8\cos\left(\frac{\pi x}{a}\right)\right)\exp(-4\omega t)\sin\left(\frac{\pi x}{a}\right)$$

$$\int_0^a \left(\frac{\partial^2 \Psi}{\partial x^2}\right)^2 \mathrm{d}x = \frac{\pi^4}{a^5}\left(\frac{a}{2}\exp(6\omega t) + 8a\right)\exp(-8\omega t)$$

$$\int_0^a \Psi \frac{\partial^2 \Psi}{\partial x^2}\,\mathrm{d}x = -\frac{\pi^2}{a^3}\left(\frac{a}{2}\exp(6\omega t) + 2a\right)\exp(-8\omega t)$$

plugging back into σ_H^2 and simplifying with a CAS, we get

$$\sigma_H^2 = \frac{\pi^4 \hbar^4}{16a^4 m^2}\big(25\exp(8\omega t) - 8\exp(6\omega t) - 8\big)\exp(-8\omega t)$$

Finally, we subsitute σ_H and σ_x into the energy-time uncertainty principle

$$\Delta E = \sigma_H = \sqrt{\frac{\pi^4 \hbar^4}{16a^4 m^2}\big(25\exp(8\omega t) - 8\exp(6\omega t) - 8\big)\exp(-8\omega t)}$$

$$= \frac{\pi^2 \hbar^2}{4a^2 m}\sqrt{\big(25\exp(8\omega t) - 8\exp(6\omega t) - 8\big)\exp(-8\omega t)}$$

$$\Delta t = \frac{\sigma_x}{|\mathrm{d}\langle x \rangle / \mathrm{d}t|} = a\sqrt{\frac{1}{12} - \frac{5}{16\pi^2} + \left(\frac{16}{9\pi^2}\cos(3\omega t)\right)^2}\,\frac{3ma}{8\hbar}\frac{1}{\sin(3\omega t)}$$

We want to show that $\Delta E \Delta t \geq \hbar/2$, but I've run out of time to show this. Hopefully taking that product would simplify to something which can be shown to be at least $\hbar/2$.

Problem 3.31 Virial theorem. Use Equation 3.71 to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle xp \rangle = 2\langle T \rangle - \left\langle x\frac{\mathrm{d}V}{\mathrm{d}x} \right\rangle,$$

where T is the kinetic energy ($H = T + V$). In a *stationary* state the left side is zero (why?) so

$$2\langle T \rangle = \left\langle x\frac{\mathrm{d}V}{\mathrm{d}x} \right\rangle.$$

This is called the **virial theorem**. Use it to prove that $\langle T \rangle = \langle V \rangle$ for stationary states of the harmonic oscillator, and check that this is consistent with the results you got in Problems 2.11 and 2.12.

Equation 3.71 states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q \rangle = \frac{\imath}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

By inspecting the equation we are seeking to solve, $Q = xp$, and $\hat{Q} = x\hat{p}$. We will begin by expanding $[\hat{H}, x\hat{p}]$. First, recall that

$$\hat{p} = \frac{\hbar}{\imath}\frac{\partial}{\partial x} \text{ and } \hat{H} = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x).$$

By the definition of the commutator, it follows that

$$\begin{aligned}
[\hat{H}, x\hat{p}]\,(f) &= \hat{H}x\hat{p}(f) - x\hat{p}\hat{H}(f) \\
&= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)x\left(\frac{\hbar}{\imath}\frac{\partial}{\partial x}\right)(f) - x\left(\frac{\hbar}{\imath}\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)(f) \\
&= \frac{\hbar}{\imath}\left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)x\left(\frac{\partial f}{\partial x}\right) - x\left(\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial x^2} + V(x)f(x)\right)\right] \\
&= \frac{\hbar}{\imath}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(x\frac{\partial f}{\partial x}\right) + xV(x)\frac{\partial f}{\partial x} + x\frac{\hbar^2}{2m}\frac{\partial^3 f}{\partial x^3} - x\frac{\partial}{\partial x}\big(V(x)f(x)\big)\right] \\
&= \frac{\hbar}{\imath}\left[-\frac{\hbar^2}{2m}\left(2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^3 f}{\partial x^3}\right) + xV(x)\frac{\partial f}{\partial x} + x\frac{\hbar^2}{2m}\frac{\partial^3 f}{\partial x^3} - x\frac{\partial V}{\partial x}f(x) - xV(x)\frac{\partial f}{\partial x}\right] \\
&= \frac{\hbar}{\imath}\left[-\frac{\hbar^2}{m}\frac{\partial^2 f}{\partial x^2} - x\frac{\partial V}{\partial x}f(x)\right]
\end{aligned}$$

now we remove the test

$$[\hat{H}, x\hat{p}] = \frac{\hbar}{\imath}\left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2} - x\frac{\partial V}{\partial x}\right]$$

and substitute back into Equation 3.71

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle xp \rangle &= \frac{\imath}{\hbar}\left\langle -\frac{\hbar}{\imath}\left(\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2} + x\frac{\partial V}{\partial x}\right)\right\rangle + \left\langle \frac{\partial(x\hat{p})}{\partial t}\right\rangle \\
&= \left\langle -\frac{\hbar^2}{m}\frac{\partial^2}{\partial x^2}\right\rangle - \left\langle x\frac{\partial V}{\partial x}\right\rangle + \left\langle \frac{\partial(x\hat{p})}{\partial t}\right\rangle \\
&= 2\langle T \rangle - \left\langle x\frac{\partial V}{\partial x}\right\rangle + \left\langle \frac{\partial(x\hat{p})}{\partial t}\right\rangle
\end{aligned}$$

In the typical case, the operator $\hat{Q}$ does not depend on time, and therefore we can neglect the time derivative of $x\hat{p}$, leaving us with the desired result:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle xp \rangle = 2\langle T \rangle - \left\langle x\frac{\partial V}{\partial x}\right\rangle.$$

In a stationary state, *nothing* depends on time, and therefore the time derivative of the expectation value of xp is zero, leaving us with

$$2\langle T \rangle = \left\langle x\frac{\partial V}{\partial x}\right\rangle.$$

The harmonic oscillator has potential $V(x) = m\omega^2 x^2/2$, and so $\frac{\mathrm{d}V}{\mathrm{d}x} = m\omega^2 x$, meaning $\left\langle x\frac{\mathrm{d}V}{\mathrm{d}x}\right\rangle = \langle m\omega^2x^2\rangle = \langle 2V\rangle$, therefore $2\langle T\rangle = 2\langle V\rangle$, or simply $\langle T\rangle = \langle V\rangle$.

Problem 3.38 The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Two other observables, A and B, are represented by the matrices

$$\mathbf{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathbf{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω, λ, and μ are positive real numbers.

a. Find the eigenvalues and (normalized) eigenvectors of **H**, **A**, and **B**.

(Note that all eigenvalues and eigenvectors were found with the assistance of `numpy.linalg.eig`)

H has eigenvalues $E_1 = \hbar\omega$, $E_2 = E_3 = 2\hbar\omega$, and eigenvectors $|H_1\rangle = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $|H_2\rangle = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, and $|H_3\rangle = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$.

A has eigenvalues $a_1 = \lambda$, $a_2 = -\lambda$, $a_3 = 2\lambda$, and eigenvectors $|A_1\rangle = (1/\sqrt{2})\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$, $|A_2\rangle = (1/\sqrt{2})\left(\begin{smallmatrix}-1\\1\\0\end{smallmatrix}\right)$, $|A_3\rangle = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$.

B has eigenvalues $b_1 = \mu$, $b_2 = -\mu$, $b_3 = 2\mu$, and eigenvectors $|B_1\rangle = (1/\sqrt{2})\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$, $|B_2\rangle = (1/\sqrt{2})\left(\begin{smallmatrix}0\\1\\-1\end{smallmatrix}\right)$, $|B_3\rangle = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$.

b. Suppose the system starts out in the generic state

$$|\mathcal{S}(0)\rangle = \left(\begin{smallmatrix}c_1\\c_2\\c_3\end{smallmatrix}\right),$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. Find the expectation values (at $t = 0$) of H, A, and B.

$$\begin{aligned}
\langle H\rangle = \langle\mathcal{S}(0)|\, H\,|\mathcal{S}(0)\rangle &= (\, c_1^* \; c_2^* \; c_3^*\,)\hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \left(\begin{smallmatrix}c_1\\c_2\\c_3\end{smallmatrix}\right) \\
&= \hbar\omega(\, c_1^* \; c_2^* \; c_3^*\,)\left(\begin{smallmatrix}c_1\\2c_2\\2c_3\end{smallmatrix}\right) = \hbar\omega\Big(|c_1|^2 + 2\big(|c_2|^2 + |c_3|^2\big)\Big) \\
\text{recall: } 1 = |c_1|^2 + |c_2|^2 + |c_3|^2 &\implies 1 - |c_1|^2 = |c_2|^2 + |c_3|^2 \\
\langle H\rangle = \hbar\omega\Big(|c_1|^2 + 2\big(1 - |c_1|^2\big)\Big) &= \hbar\omega\big(2 - |c_1|^2\big), \\
\langle A\rangle = \langle\mathcal{S}(0)|\, A\,|\mathcal{S}(0)\rangle &= (\, c_1^* \; c_2^* \; c_3^*\,)\lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \left(\begin{smallmatrix}c_1\\c_2\\c_3\end{smallmatrix}\right) \\
&= \lambda(\, c_1^* \; c_2^* \; c_3^*\,)\left(\begin{smallmatrix}c_2\\c_1\\2c_3\end{smallmatrix}\right) = \lambda\big(c_1^*c_2 + c_2^*c_1 + 2|c_3|^2\big), \\
\langle B\rangle = \langle\mathcal{S}(0)|\, B\,|\mathcal{S}(0)\rangle &= (\, c_1^* \; c_2^* \; c_3^*\,)\mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{smallmatrix}c_1\\c_2\\c_3\end{smallmatrix}\right) \\
&= \mu(\, c_1^* \; c_2^* \; c_3^*\,)\left(\begin{smallmatrix}2c_1\\c_3\\c_2\end{smallmatrix}\right) = \mu\big(2|c_1|^2 + c_2^*c_3 + c_3^*c_2\big).
\end{aligned}$$

c. What is $|\mathcal{S}(t)\rangle$? If you measured the energy of this state (at time t), what values might you get, and what is the probability of each? Answer the same questions for A and for B.

We obtain $|\mathcal{S}(t)\rangle$ by writing $|\mathcal{S}(0)\rangle$ as a linear combination of time-independent eigenstates, and then tacking on the time dependence, $\exp(-\imath E_n t/\hbar)$.

One can intuitively write $|\mathcal{S}(0)\rangle$ as a linear combination of H's eigenvectors, given by

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} = c_1 |H_1\rangle + c_2 |H_2\rangle + c_3 |H_3\rangle .$$

Then by tacking on the time-dependence we obtain $|\mathcal{S}(t)\rangle$

$$|\mathcal{S}(t)\rangle = c_1 |H_1\rangle \exp(-\imath E_1 t/\hbar) + c_2 |H_2\rangle \exp(-\imath E_2 t/\hbar) + c_3 |H_3\rangle \exp(-\imath E_3 t/\hbar).$$

If you were to measure the energy of the state, there is a probability $|c_1|^2$ that you would observe $E_1 = \hbar\omega$, and a probability $|c_2|^2 + |c_3|^2$ that you would observe $E_2 = E_3 = 2\hbar\omega$.