Quantum Mechanics – Chapter 2

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The Free Particle

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$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -k^2\psi, \quad \text{where } k := \frac{\sqrt{2mE}}{\hbar}$$

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$$\Psi(x,t) = A \exp\left[\imath k \left(x - \frac{\hbar k}{2m} t\right)\right] + B \exp\left[-\imath k \left(x + \frac{\hbar k}{2m} t\right)\right]$$

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- the first term represents a wave travelling to the right, and the second to the left

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- now we may rewrite the wave function as

$$\Psi_k(x, t) = A \exp\left[\imath\left(kx - \frac{\hbar k^2}{2m} t\right)\right]$$

Normalization

- we cannot normalize Ψ_k, because $\Psi_k^* \Psi_k = |A|^2$, giving

$$\int_{-\infty}^{+\infty} \Psi_k^* \Psi_k \, \mathrm{d}x = |A|^2 \int_{-\infty}^{+\infty} \mathrm{d}x = |A|^2 \cdot \infty$$

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- in essence, $(1/\sqrt{2\pi})\phi(k)\,\mathrm{d}k$ is taking the place of the coefficients c_n in the discrete summation

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- we only now have to solve for $\phi(k)$

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General Solution

- this is a classic problem in Fourier analysis, whose answer is provided by **Plancherel's theorem**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k) e^{\imath kx} \, \mathrm{d}k \iff F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-\imath kx} \, \mathrm{d}x$$

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- now we can find $\Psi(x, t)$

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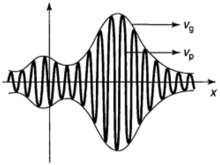
$$v_{\text{quantum}} = \frac{\hbar |k|}{2m} = \sqrt{\frac{E}{2m}}$$

- this is contrary to *classical* speed, which can be determined, for a free particle, by kinetic energy $E = (1/2)mv^2$

$$v_{\text{classical}} = \sqrt{\frac{2E}{m}} = 2v_{\text{quantum}}$$

Group and Phase Velocity



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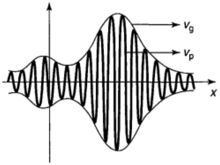

- the quantum velocity corresponds to the *phase velocity*, the velocity of the individual ripples
- the classical velocity corresponds to the *group velocity*, the velocity of the *envelope*

The Delta-Function Potential

Classical Bound States and Scattering States

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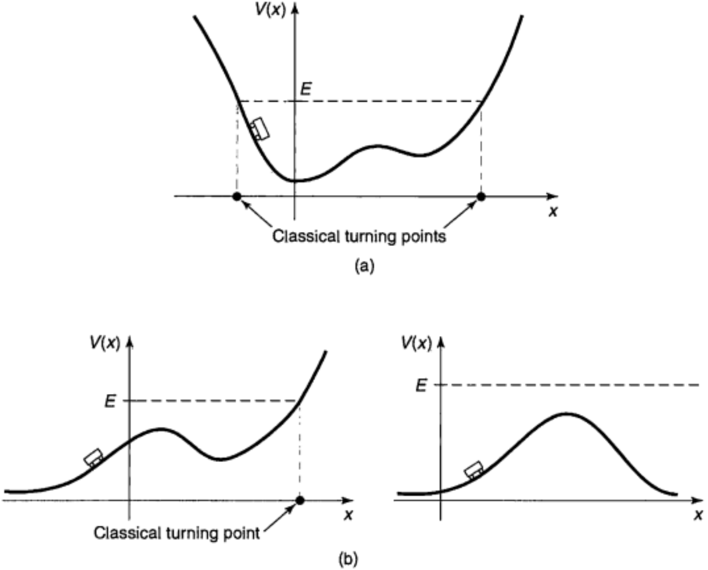

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- in practice, most potentials go to *zero* at infinity, simplifying the criterion to

$$\begin{cases} E < 0 \implies & \text{bound state,} \\ E > 0 \implies & \text{scattering state.} \end{cases}$$

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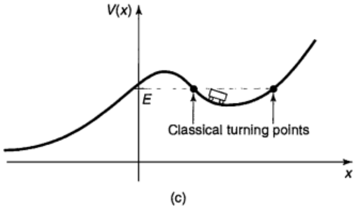
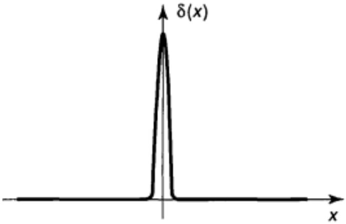


- *bound* state for classical particle, but *scattering* state for quantum particle

The Delta–Function Well



- the **Dirac delta function** has infinite height, infinitesimal width, and an *area* of 1

$$\delta(x) := \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}, \quad \text{with } \int_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x = 1.$$

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- multiplying by a function $f(x)$ is equivalent to multiplying by $f(a)$, as it is zero everywhere outside of a

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Reflection and Transmission

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- the probability of transmission is proportional to the energy

The Finite Square Well

Problem

- consider the *finite* square well potential, where V_0 is a positive real potential

$$V(x) = \begin{cases} -V_0, & \text{for } -a \leq x \leq a, \\ 0, & \text{for } |x| > a, \end{cases}$$

General Solution

- the general solution is given by

$$\begin{cases} Fe^{-\kappa x}, & \text{for } x > a, \\ D\cos(lx), & \text{for } 0 < x < a, \\ \psi(-x), & \text{for } x < 0. \end{cases}$$

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- continuity of $\psi(x)$ and $\frac{\mathrm{d}\psi}{\mathrm{d}x}$ at the boundaries imply $\kappa = l\tan(la)$, where

$$\kappa := \frac{\sqrt{-2mE}}{\hbar}$$
$$l := \frac{\sqrt{2m(E + V_0)}}{\hbar}$$

Energy of the Finite Square Well

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- can only be solved numerically

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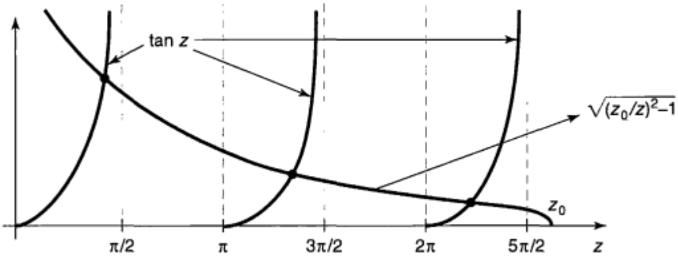

Wide, Deep Well

- if z_0 is very large, the intersections occur just below $z_n = n\pi/2$, where n is odd

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- there are a finite number of bound states, but as $V_0 \to \infty$, it approaches the infinite square well, with infinite bound states

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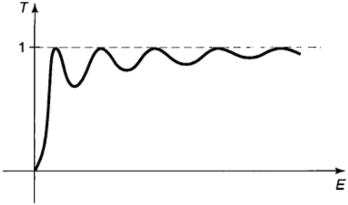
- this reaches a limit at $z_0 < \pi/2$, where the lowest *odd* state disappears, leaving a single state
- no matter how small z_0 becomes, the number of bound states is always at least one

Transmission



$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

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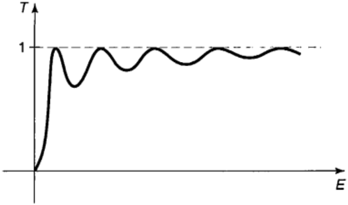

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- these are the allowed energies of the infinite square well

Thank you!